

Scaling algebras for charged fields and short-distance analysis for localizable and topological charges

CLAUDIO D'ANTONI ^{*1}

GERARDO MORSELLA ^{** 1}

RAINER VERCH ^{***}

^{*} Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica, I-00133 Roma, Italy — e-mail: dantoni@mat.uniroma2.it

^{**} Dipartimento di Matematica, Università di Roma “La Sapienza”, P.le Aldo Moro 2, I-00185 Roma, Italy — e-mail: morsella@mat.uniroma1.it

^{***} Max-Planck-Institut für Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany — e-mail: verch@mis.mpg.de

Abstract. The method of scaling algebras, which has been introduced earlier as a means for analyzing the short-distance behaviour of quantum field theories in the setting of the model-independent, operator algebraic approach, is extended to the case of fields carrying superselection charges. In doing so, consideration will be given to strictly localizable charges (“DHR-type” superselection charges) as well as to charges which can only be localized in regions extending to spacelike infinity (“BF-type” superselection charges). A criterion for the preservice of superselection charges in the short-distance scaling limit is proposed. Consequences of this preservice of superselection charges are studied. The conjugate charge of a preserved charge is also preserved, and for charges of DHR-type, the preservice of all charges of a quantum field theory in the scaling limit leads to equivalence of local and global intertwiners between superselection sectors.

1 Introduction

In an attempt to analyze the short-distance behaviour of quantum field theories in a completely model-independent manner, and to have a counterpart of renormalization group analysis at short length scales in the setting of general quantum field theory, so-called “scaling algebras” have been introduced some time ago [10]. The idea of this approach is to associate to a given quantum field theory described in terms of local observable algebras [21, 20] a “scaling algebra” of functions depending on a scaling parameter $\lambda > 0$ and taking values in the local observable algebras. These functions are required to have certain

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properties regarding their localization and energy behaviour as λ tends to zero; roughly speaking, the values of the functions at scale parameter λ should be observables localized in spacetime regions of extension proportional to λ , and having energy-momentum transfer proportional to λ^{-1} .

The collection of all these functions, i.e. of all the members of the scaling algebra, may hence be viewed as “orbits” of elements in the local observable algebras under all possible renormalization group transformations. By studying the vacuum expectation values of these functions in the limit $\lambda \rightarrow 0$ (the “scaling limit”), one can then analyze the extreme short distance properties of the given quantum field theory.

This programme, initiated in [10], has been further developed in [8, 11, 7, 24]. It leads to a general classification of the short distance behaviour of the given theory which corresponds to the one known in perturbation theory where one distinguishes theories with stable ultraviolet fixed points under renormalization group transformations, as opposed to others with unstable fixed points or no fixed points at all [10].

Moreover, it permits to give a criterion as to when a given quantum field theory possesses “confined charges” which are only visible in the extreme short distance limit while they are absent at finite scale, like the colour charge in QCD [6, 8]. According to this criterion a charge is confined if it arises as a superselection charge in the scaling limit theory of the observables which is not a scaling limit of the superselection charges of the original theory at scale $\lambda = 1$ (see Sec. 5 for discussion). The effectiveness of this criterion has been illustrated in the example of the two-dimensional Schwinger model [8, 11].

However, with the exception of ref. [25], the scaling algebra method has up to now only been applied in the setting of local observable algebras, not in the context of local field algebras containing charge-carrying local field algebras. In other words, this method has not yet been applied to studying the short-distance behaviour of superselection charges (see [30, 20] and references cited there) and their corresponding charge-carrying fields.

In the present work, we generalize the “scaling algebra” framework in the setting of algebraic quantum field theory in the presence of field operators transforming non-trivially under the action of a (global) compact gauge group. We consider separately two cases where the field operators can be localized (1) in bounded spacetime regions and (2) in regions extending to spacelike infinity (so-called “spacelike cones”). The first case corresponds to superselection charges which can be localized in arbitrary bounded regions of spacetime (“DHR-charges”) while the second case corresponds to superselection sectors carrying so-called topological charges (“BF-charges”).² In both cases, we will assume that the translations act covariantly on the algebras of field operators, and that there is a translation-invariant vacuum. Our principal interest lies in the behaviour of the superselection charges in the scaling limit.

We propose a criterion specifying what it means that a charge superselection sector of the given quantum field theory is “preserved” in the scaling limit. Then we will show that under quite general conditions, a superselection charge is preserved in the scaling limit exactly if this is also the case for the corresponding conjugate charge. As a further application, we extend an earlier result by Roberts [29] (which was obtained for dilation covariant quantum field theories) by showing that in a quantum field theory where all

²The acronyms DHR and BF refer to Doplicher-Haag-Roberts [12, 13] and to Buchholz-Fredenhagen [9], respectively, who have introduced and analyzed the corresponding types of superselection sectors.

charges of DHR-type are preserved in the scaling limit, the sets of local and global intertwiners for the superselection charges coincide (see the first part of Sec. 4 for explanation of this terminology). This amounts to saying that part of the superselection structure is determined locally if the superselection charges are ultraviolet stable in the sense of being preserved in the scaling limit. Such a property is of some relevance in the construction of superselection theory in a generally covariant setting as recently developed in [36].

This article is organized as follows. In Sec. 2 we define the quantum field theories corresponding to case (1), with strictly localizable field operators, more precisely. We introduce a class of theories which we call “quantum field theories with gauge group action”, abbreviated QFTGA, in the operator-algebraic setting. This class of theories is slightly more general than the class of theories obtained via the Doplicher-Roberts reconstruction from DHR-type superselection charges (which will be considered in Sec. 4). We introduce the scaling algebra for such QFTGAs, and, in close analogy to [10], we introduce scaling limit states and scaling limit theories and study their basic properties.

Then, in Sec. 3, we consider QFTGAs with more structure, mainly with additional Poincaré covariance and clustering properties, and study what additional properties ensue in the scaling limit.

In Sec. 4 we introduce “quantum field systems with gauge symmetry” (QFSGSs) according to [15]. These are more special QFTGAs which arise by the Doplicher-Roberts reconstruction theorem from the covariant, strictly localizable (i.e. DHR-type) superselection sectors with finite statistics belonging to a quantum field theory of local observables (cf. again [15]). Charges of this kind would, e.g., correspond to the flavour charges of strong interactions. The reason why we make a distinction between QFTGA and QFSGS is that the scaling limit theories of a QFTGA are again of this type, i.e. are QFTGAs. But scaling limit theories of a QFSGS have in general only the structure of a QFTGA. We summarize parts of the terminology of the theory of superselection sectors and the result on the existence of a corresponding QFSGS, emphasizing the role played by the “field multiplets” in the local field algebras corresponding to each superselection charge.

We will make use of this in Sec. 5, where we will state our criterion of preservice of a charge in the scaling limit in terms of such field multiplets: Our criterion demands that a charge of DHR-type is preserved in the scaling limit if scaled families of such multiplets (“scaled multiplets”) have a certain limiting behaviour in the scaling limit. Then we briefly discuss mechanisms for the disappearance of charges in the scaling limit. Quite generally, a charge may disappear in the scaling limit if it takes typically more energy than proportional to λ^{-1} to create the charge within a spacetime region of extension proportional to λ . As will be explained, this may happen if the interaction between the charges of a quantum field theory is, at extremely short distances, either strongly binding or strongly repellent. Moreover, we present some further results on the structure of superselection charges preserved in the scaling limit, like the preservice of the conjugate charge.

In Sec. 6 we state and prove our result on the equivalence of local and global intertwiners if all DHR-charges are preserved in the scaling limit.

Having up to this point discussed the case (1) of field operators localizable in arbitrary bounded regions, we will turn in Sec. 7 to the discussion of QFSGSs where the field operators are only localizable in infinitely extended spacelike cones (corresponding to

case (2) alluded to above). Because of the much weaker localization properties of field operators in this case, the scaling algebra method has to be appropriately adapted, and this is done by defining the scaling algebra functions of field operators such that the functions asymptotically (as the scaling parameter λ tends to 0) commute with the scaling algebra functions of the observables (which are strictly localized) from which the QFSGS is constructed by the Doplicher-Roberts theorem for superselection charges of the BF-type. It will turn out that, for the corresponding definition of the scaling algebra of fields, one obtains scaling limit QFTGAs where the field operators are strictly localizable.

In Sec. 8 we will then generalize the criterion for the preservice of charges in the scaling limit to the case of BF-type superselection charges, and discuss conditions sufficient to conclude that such preserved charges induce superselection sectors of the scaling limit theories of the observables. It will be assumed in this discussion that the underlying QFSGS of fields localized in spacelike cones is Poincaré covariant.

There will be two more technical Appendices. In Appendix A, we present the example of the Majorana-Dirac field which possesses a \mathbb{Z}_2 gauge symmetry, and show that the corresponding charges are preserved in the scaling limit according to our criterion. In Appendix B, we sketch the proof of a Reeh-Schlieder property for an extended scaling limit field algebra used in the construction of the scaling limit QFTGA in Sec. 7.

The article is completed by some concluding remarks.

2 Quantum field theories with gauge group action and their scaling algebras and scaling limits

In the present section we investigate an extension of the “scaling algebra” approach of [10] to quantum field theories that include a structure which we will call a *normal, covariant quantum field theory with gauge group action* (QFTGA) since we will see that this structure has a counterpart in the scaling limit. In the next section we add a few more assumptions, such as Lorentz covariance, geometric modular action, and clustering, but it is not before Section 4 that we introduce a *normal, covariant quantum field system with gauge symmetry* according to [15] which connects quantum field algebras and superselection sectors, and explore some properties of the scaling limits for such theories.

Notation. In the following, we consider quantum field theories on n -dimensional Minkowski-spacetime ($n \geq 2$), which will be identified with \mathbb{R}^n , equipped with the Lorentzian metric $\eta = (\eta_{\mu\nu}) = \text{diag}(1, -1, -1, \dots, -1)$. We recall that the set $V_+ := \{(y^0, \dots, y^{n-1}) \in \mathbb{R}^n : (y^0)^2 > (y^1)^2 + \dots + (y^{n-1})^2, y^0 > 0\}$ denotes the open forward lightcone and \overline{V}_+ its closure. A *double cone* is any set in \mathbb{R}^n of the form $O = x + V_+ \cap y - V_+$ for any pair of $x, y \in \mathbb{R}^n$ so that $y \in x + V_+$. The set of all double cones in \mathbb{R}^n will be denoted generically by \mathcal{K} .

Definition 2.1. A quintuple $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$ is called a *normal, covariant quantum field theory with gauge group action* (QFTGA) if the following properties are fulfilled:

(QFTGA.1) There is a Hilbert-space \mathcal{H} and a family $\{\mathcal{F}(O)\}_{O \in \mathcal{K}}$ of von Neumann algebras on \mathcal{H} which is indexed by the members O of the set \mathcal{K} of all double cones in

n -dimensional Minkowski spacetime. It will be assumed that isotony holds, i.e.

$$O_1 \subset O \Rightarrow \mathcal{F}(O_1) \subset \mathcal{F}(O).$$

Hence, one may form the smallest C^* -algebra $\mathfrak{F} := \overline{\bigcup_O \mathcal{F}(O)}^{C^*}$ in $B(\mathcal{H})$ containing all local field algebras $\mathcal{F}(O)$. (In the above quintuple, \mathcal{F} is short for the family $\{\mathcal{F}(O)\}_{O \in \mathcal{K}}$.)

(QFTGA.2) There is a strongly continuous unitary representation $\mathbb{R}^n \ni a \mapsto \mathcal{U}(a) \in B(\mathcal{H})$ of the group of translations \mathbb{R}^n on \mathcal{H} whose action on $\{\mathcal{F}(O)\}_{O \in \mathcal{K}}$ is covariant, i.e.

$$\mathcal{U}(a)\mathcal{F}(O)\mathcal{U}(a)^* = \mathcal{F}(O + a), \quad a \in \mathbb{R}^n, \quad O \in \mathcal{K}.$$

Moreover, it will be assumed that the relativistic spectrum condition holds: The joint spectrum of the selfadjoint generators of $\mathcal{U}(\mathbb{R}^n)$ is contained in the closed forward lightcone $\overline{V^+}$.

(QFTGA.3) There is a compact group G , and a strongly continuous,³ faithful representation $G \ni g \mapsto U(g) \in B(\mathcal{H})$ of the group G on \mathcal{H} . It is assumed that the action of this unitary representation on $\{\mathcal{F}(O)\}_{O \in \mathcal{K}}$ preserves localization, i.e.

$$U(g)\mathcal{F}(O)U(g)^* = \mathcal{F}(O), \quad g \in G, \quad O \in \mathcal{K},$$

and also that this group representation commutes with the translations:

$$U(g)\mathcal{U}(a) = \mathcal{U}(a)U(g), \quad g \in G, \quad a \in \mathbb{R}^n.$$

G will be called the *gauge group*.

(QFTGA.4) There is a unit vector $\Omega \in \mathcal{H}$ which is invariant under all $\mathcal{U}(a)$, $a \in \mathbb{R}^n$, and under all $U(g)$, $g \in G$, and which moreover has the cyclicity property $\overline{\mathfrak{F}\Omega} = \mathcal{H}$. This vector is called the *vacuum vector*.

(QFTGA.5) There is an element k contained in the centre of G and fulfilling $k^2 = 1_G$ (the unit group element) so that, upon setting

$$F_{\pm} := \frac{1}{2}(F \pm U(k)FU(k)^*),$$

the following relations hold whenever $F \in \mathcal{F}(O_1)$, $F' \in \mathcal{F}(O_2)$, and the double cones O_1 and O_2 are spacelike sparated:

$$F_+F'_+ = F'_+F_+, \quad F_+F'_- = F'_-F_+, \quad F_-F'_- = -F'_-F_- . \quad (2.1)$$

These properties are referred to as *normal commutation relations*.

³whenever this makes sense, i.e. when G possesses continuous parts

Remark. It was already mentioned in the introduction that the definition of a QFTGA is slightly more general than that of a quantum field system with gauge symmetry (see Sec. 4) which is more directly related to the theory of superselection charges; however, the differences are minute and mainly of technical nature. The advantage of working with QFTGAs is that their structure is stable with respect to passing to scaling limit theories, as will become clear in the present section.

The next task is to introduce the counterpart of the scaling algebra for a QFTGA which was defined in [10] for quantum field theories formulated in terms of local observable algebras. To that end, we assume that we are given an arbitrary normal, covariant quantum field theory with gauge group action $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$ (henceforth called the “underlying QFTGA”) and keep it fixed. It will be convenient to introduce the following notation for the adjoint actions of translations and gauge group:

$$\alpha_a(F) := \mathcal{U}(a)F\mathcal{U}(a)^*, \quad \beta_g(F) := U(g)FU(g)^*,$$

for all $F \in \mathfrak{F}$, $a \in \mathbb{R}^n$, $g \in G$.

Definition 2.2. For each $O \in \mathcal{K}$, we define $\underline{\mathfrak{F}}(O)$ as the set of all functions $\underline{F} : \mathbb{R}^+ \rightarrow \mathfrak{F}$, $\lambda \mapsto \underline{F}_\lambda$, having the following properties:

- (a) $\underline{F}_\lambda \in \mathcal{F}(\lambda O)$,
- (b) $\|\underline{F}\| := \sup_\lambda \|\underline{F}_\lambda\| < \infty$,
- (c) $\|\underline{\alpha}_a(\underline{F}) - \underline{F}\| \rightarrow 0$ as $a \rightarrow 0$, where

$$(\underline{\alpha}_a(\underline{F}))_\lambda := \alpha_{\lambda a}(\underline{F}_\lambda), \tag{2.2}$$

- (d) $\|\underline{\beta}_g(\underline{F}) - \underline{F}\| \rightarrow 0$ as $g \rightarrow 1_G$, where

$$(\underline{\beta}_g(\underline{F}))_\lambda := \beta_g(\underline{F}_\lambda). \tag{2.3}$$

In [10] the case was considered that \mathfrak{F} is an observable algebra. In that case, the action of the gauge group $U(G)$ on \mathcal{H} is trivial, and spacelike commutativity holds for the local algebras $\mathcal{F}(O)$, meaning that $\mathcal{F}(O_1) \subset \mathcal{F}(O_2)'$ if O_1 and O_2 are spacelike separated. The motivation for imposing the conditions (a-d) above is similar as for the scaling algebra in the case that \mathcal{F} is an observable algebra discussed in [10]. The idea is to view the \underline{F}_λ as the image of an element $F \in \mathfrak{F}$ under the action of any “renormalization group transformation” R_λ (so one should think of \underline{F}_λ as $R_\lambda(F)$). In other words, the collection of all functions $\lambda \mapsto \underline{F}_\lambda$ with the above stated properties corresponds to all possible orbits of elements in \mathfrak{F} under all (abstract) renormalization group transformations. The general properties of renormalization group transformations in the present, model-independent setting are hence encoded by the conditions (a-d). We point out that (c) ensures that the energy-momentum transferred by \underline{F}_λ scales like $\text{const.} \cdot 1/\lambda$, see [10] for further discussion.

As has been indicated to us by D. Buchholz, it should be noted that there may actually be situations where the lifted action of the gauge transformations ought to be

defined differently than in (2.3). This occurs for example if the charges of the theory have a dimension which isn't independent of length or energy (in this sense, they are “dimensionful” charges), and this can happen in two-dimensional models. For the time being, we neglect this possibility, but we point out that it deserves attention.

There are some simple consequences of Def. 2.2 which we briefly put on record here, see [10] for more details. First, it is easy to see that each $\underline{\mathfrak{F}}(O)$, $O \in \mathcal{K}$, is a C^* -algebra with respect to the C^* -norm introduced in (b) when the algebraic operations are defined pointwise for each λ . Clearly one also has isotony,

$$O_1 \subset O \Rightarrow \underline{\mathfrak{F}}(O_1) \subset \underline{\mathfrak{F}}(O).$$

One can thus form the C^* -algebra $\underline{\mathfrak{F}} = \overline{\bigcup_O \underline{\mathfrak{F}}(O)}^{C^*}$. The “lifted” actions $\underline{\alpha}_{\mathbb{R}^n}$ and $\underline{\beta}_G$ of translations and gauge group, defined in (2.2) and (2.3), respectively, act by automorphisms on $\underline{\mathfrak{F}}$ under preservation of the corresponding covariance properties, i.e.

$$\underline{\alpha}_a(\underline{\mathfrak{F}}(O)) = \underline{\mathfrak{F}}(O + a), \quad \underline{\beta}_g(\underline{\mathfrak{F}}(O)) = \underline{\mathfrak{F}}(O). \quad (2.4)$$

Moreover, we may define

$$\underline{F}_{\pm} = \frac{1}{2}(\underline{F} \pm \underline{\beta}_k(\underline{F}))$$

and hence obtain relations similar to (2.1) for $\underline{F} \in \underline{\mathfrak{F}}(O_1)$, $\underline{F}' \in \underline{\mathfrak{F}}(O_2)$ and O_1 and O_2 spacelike separated. Finally we note that one may demonstrate the existence of a wealth of elements in $\underline{\mathfrak{F}}$ as follows. Let μ be a left-invariant Borel-measure on G and let h be any continuous, compactly supported function on $\mathbb{R}^d \times G$. Pick any uniformly bounded function $\mathbb{R}^+ \ni \lambda \mapsto X_\lambda \in \mathfrak{F}$ so that $X_\lambda \in \mathcal{F}(\lambda O)$ for each λ and some $O \in \mathcal{K}$, and define

$$\underline{F}_\lambda := \int d^n a \, d\mu(g) \, h(a, g) \, \alpha_{\lambda a}(\beta_g(X_\lambda)) \quad (2.5)$$

where the integral is to be understood in the weak sense. Then it is easily checked that $\mathbb{R}^+ \ni \lambda \mapsto \underline{F}_\lambda$ is contained in $\underline{\mathfrak{F}}(O^\times)$ whenever O^\times is any open neighbourhood of $\overline{O} + \bigcup_{g \in G} \text{supp } h(\cdot, g)$.

Having defined the scaling field algebra $\underline{\mathfrak{F}}$ of the underlying QFTGA, we may associate with any locally normal state ω' on \mathfrak{F} ⁴ a parametrized family $(\underline{\omega}'_\lambda)_{\lambda>0}$ of states on $\underline{\mathfrak{F}}$, where

$$\underline{\omega}'_\lambda(\underline{F}) := \omega'(\underline{F}_\lambda), \quad \underline{F} \in \underline{\mathfrak{F}}.$$

As in [10], we adopt the following definition of scaling limit states.

Definition 2.3. For each locally normal state ω' on \mathfrak{F} , we regard the family $(\underline{\omega}'_\lambda)_{\lambda>0}$ as a generalized sequence directed towards $\lambda = 0$. Hence, by the Banach-Alaoglu theorem [27], the family $(\underline{\omega}'_\lambda)_{\lambda>0}$ on the C^* -algebra $\underline{\mathfrak{F}}$ possesses weak-* limit points. This set of weak-* limit points will be denoted by $\{\omega'_{0,\iota} : \iota \in \mathbb{I}\}$ where \mathbb{I} is a suitable index set, or simply by $\text{SL}^{\mathfrak{F}}(\omega')$. Each $\omega'_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega')$ is a state on $\underline{\mathfrak{F}}$, and is called a *scaling limit state* of ω' .

⁴a state ω' on \mathfrak{F} is called *locally normal* if $\omega' \upharpoonright \mathcal{F}(O)$ is normal for each $O \in \mathcal{K}$

We note that the definition of weak-* limit points means that there exists for each label ι a directed set K_ι together with a generalized sequence $(\lambda_\kappa^{(\iota)})_{\kappa \in K_\iota}$ of positive numbers converging to 0 so that

$$\omega'_{0,\iota}(\underline{F}) = \lim_{\kappa} \underline{\omega}'_{\lambda_\kappa^{(\iota)}}(\underline{F}), \quad \underline{F} \in \underline{\mathfrak{F}}.$$

Again following [10], we introduce for each scaling limit state $\omega'_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega')$ its GNS-representation $(\pi_{0,\iota}, \mathcal{H}_{0,\iota}, \Omega_{0,\iota})$ and define

$$\mathcal{F}_{0,\iota}(O) := \pi_{0,\iota}(\underline{\mathfrak{F}}(O))'', \quad \mathfrak{F}_{0,\iota} := \overline{\bigcup_O \mathcal{F}_{0,\iota}(O)}^{C^*}.$$

Many of the following results (containing also some new definitions) concerning the structure of scaling limit states and their associated GNS-representations in the present setting are generalizations of similar statements in [10].

Proposition 2.4. *1. For each pair of locally normal states ω' and ω'' on \mathfrak{F} it holds that*

$$\text{SL}^{\mathfrak{F}}(\omega') = \text{SL}^{\mathfrak{F}}(\omega'').$$

2. Let ω' be a locally normal state on \mathfrak{F} . Then each $\omega'_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega')$ is invariant under the actions of $\underline{\alpha}_a$, $a \in \mathbb{R}^n$, and $\underline{\beta}_g$, $g \in G$:

$$\omega'_{0,\iota} \circ \underline{\alpha}_a = \omega'_{0,\iota}, \quad \omega'_{0,\iota} \circ \underline{\beta}_g = \omega'_{0,\iota}.$$

Hence, there are unitary group representations of the translation group and the gauge group on $\mathcal{H}_{0,\iota}$ which are, respectively, defined by

$$\mathcal{U}_{0,\iota}(a)\pi_{0,\iota}(\underline{F})\Omega_{0,\iota} := \pi_{0,\iota}(\underline{\alpha}_a(\underline{F}))\Omega_{0,\iota}, \quad U_{0,\iota}(g)\pi_{0,\iota}(\underline{F})\Omega_{0,\iota} := \pi_{0,\iota}(\underline{\beta}_g(\underline{F}))\Omega_{0,\iota}$$

for all $a \in \mathbb{R}^n$, $g \in G$, and $\underline{F} \in \underline{\mathfrak{F}}$.

3. The unitary group representations $\mathcal{U}_{0,\iota}(a)$, $a \in \mathbb{R}^n$, and $U_{0,\iota}(g)$, $g \in G$, are continuous and have the properties

$$\mathcal{U}_{0,\iota}(a)\mathcal{F}_{0,\iota}(O)\mathcal{U}_{0,\iota}(a)^* = \mathcal{F}_{0,\iota}(O + a), \quad U_{0,\iota}(g)\mathcal{F}_{0,\iota}(O)U_{0,\iota}(g)^* = \mathcal{F}_{0,\iota}(O)$$

for all $a \in \mathbb{R}^n$, $g \in G$ and $O \in \mathcal{K}$. Moreover, the unitary translation group $\mathcal{U}_{0,\iota}(a)$, $a \in \mathbb{R}^n$, fulfills the relativistic spectrum condition.

4. The set $N_{0,\iota}$ of all $g \in G$ so that $U_{0,\iota}(g)\psi = \psi$ holds for all $\psi \in \mathcal{H}_{0,\iota}$ is a closed normal subgroup of G . Therefore,

$$U_{0,\iota}^\bullet : G_{0,\iota}^\bullet \ni g^\bullet \mapsto U_{0,\iota}(g) \tag{2.6}$$

is a continuous faithful representation of the factor group $G_{0,\iota}^\bullet = G/N_{0,\iota}$. Here, $g \mapsto g^\bullet \equiv g_{0,\iota}^\bullet$ is the quotient map, and in (2.6), g is any element in the pre-image of g^\bullet with respect to the quotient map.

5. Define for $\mathbf{f} \in \mathfrak{F}_{0,\iota}$,

$$\mathbf{f}_\pm := \frac{1}{2}(\mathbf{f} \pm U_{0,\iota}^\bullet(k^\bullet) \mathbf{f} U_{0,\iota}^\bullet(k^\bullet)^*)$$

where $k^\bullet = k_{0,\iota}^\bullet$. Then the following holds: If O_1 and O_2 are spacelike separated double cones and $\mathbf{f} \in \mathfrak{F}_{0,\iota}(O_1)$, $\mathbf{f}' \in \mathfrak{F}_{0,\iota}(O_2)$, one has the relations

$$\mathbf{f}_+ \mathbf{f}'_+ = \mathbf{f}'_+ \mathbf{f}_+, \quad \mathbf{f}_+ \mathbf{f}'_- = \mathbf{f}'_- \mathbf{f}_+, \quad \mathbf{f}_- \mathbf{f}'_- = -\mathbf{f}'_- \mathbf{f}_-. \quad (2.7)$$

6. The previous statements yield the following corollary: Let ω' be a locally normal on \mathfrak{F} (of the underlying QFTGA) and $\omega'_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega')$ an arbitrary scaling limit state, then the corresponding scaling limit objects

$(\mathfrak{F}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), U_{0,\iota}^\bullet(G_{0,\iota}^\bullet), \Omega_{0,\iota}, k_{0,\iota}^\bullet)$ form again a normal, covariant quantum field theory with gauge group action (which will be called a scaling limit QFTGA of the underlying QFTGA corresponding to $\omega'_{0,\iota}$).

Proof. Ad 1. The proof is analogous to that in [10], which uses an argument due to Roberts [29] showing that

$$\|(\omega' - \omega'') \upharpoonright \mathcal{F}(\lambda O)\| \rightarrow 0 \quad \text{as } \lambda \rightarrow 0 \quad (2.8)$$

holds for any pair of locally normal states ω' and ω'' on \mathfrak{F} and $O \in \mathcal{K}$ as a consequence of

$$\bigcap_{O \ni 0} \mathcal{F}(O) = \mathbb{C} \cdot 1.$$

This latter property holds also for the local field algebras owing to the spectrum condition for the translation group and normal commutation relations (2.1), see [10] for details.

Ad 2. The invariance property is obvious for the case that ω' coincides with the vacuum state $\omega(F) = \langle \Omega, F\Omega \rangle$ on \mathfrak{F} . Then (2.8) implies the analogous property for any other locally normal state.

Ad 3. The continuity follows simply from assumptions (c) and (d) of Def. 2.2. The covariance properties are implied by (2.4). The spectrum condition for the translations may be proved as in [10].

Ad 4. By construction, $U_{0,\iota}^\bullet$ is a faithful unitary representation of $G_{0,\iota}^\bullet$ on $\mathcal{H}_{0,\iota}$. Continuity follows since the quotient map $g \mapsto g^\bullet$ is open.

Ad 5. As indicated above, the relations (2.1) carry over to the scaling algebra \mathfrak{F} by setting $\underline{E}_\pm = \frac{1}{2}(\underline{E} \pm \underline{\beta}_k(\underline{E}))$. The corresponding relations for the scaling limit theories follow directly. (It may however happen that $k \in N_{0,\iota}$; in this case, the last, “fermionic” relation of (2.7) is absent, and spacelike commutativity holds for the local scaling limit algebras $\mathfrak{F}_{0,\iota}(O)$, $O \in \mathcal{K}$.)

Henceforth, we will (without restriction of generality in view of 1. of Prop. 2.4) always consider scaling limit states $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega)$ where $\omega(\cdot) = \langle \Omega, \cdot \Omega \rangle$ denotes the vacuum state.

As was done in [10], we will identify scaling limit theories which are isomorphic in a sense that we will describe next.

Definition 2.5. Let

$$(\mathcal{F}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), U_{0,\iota}^\bullet(G_{0,\iota}^\bullet), \Omega_{0,\iota}, k_{0,\iota}^\bullet) \quad \text{and} \quad (\mathcal{F}_{0,\gamma}, \mathcal{U}_{0,\gamma}(\mathbb{R}^n), U_{0,\gamma}^\bullet(G_{0,\gamma}^\bullet), \Omega_{0,\gamma}, k_{0,\gamma}^\bullet)$$

be two scaling limit theories of an underlying QFTGA. These two scaling limit theories will be called *isomorphic* if there exists a C^* -algebraic isomorphism $\phi : \mathfrak{F}_{0,\iota} \rightarrow \mathfrak{F}_{0,\gamma}$ so that the following properties hold:

$$\begin{aligned} \phi(\mathcal{F}_{0,\iota}(O)) &= \mathcal{F}_{0,\gamma}(O), \quad O \in \mathcal{K}, \\ \phi \circ \text{Ad } \mathcal{U}_{0,\iota}(a) &= \text{Ad } \mathcal{U}_{0,\gamma}(a) \circ \phi, \quad a \in \mathbb{R}^n, \\ \phi \circ \text{Ad } U_{0,\iota}(g) &= \text{Ad } U_{0,\gamma}(g) \circ \phi, \quad g \in G. \end{aligned}$$

Note that the last property induces a natural identification between $N_{0,\iota}$ and $N_{0,\gamma}$ and hence a natural identification $G_{0,\iota}^\bullet \ni g_{0,\iota}^\bullet \mapsto g_{0,\gamma}^\bullet \in G_{0,\gamma}^\bullet$, so that one obtains, in consequence,

$$\phi \circ \text{Ad } U_{0,\iota}^\bullet(g_{0,\iota}^\bullet) = \text{Ad } U_{0,\gamma}^\bullet(g_{0,\gamma}^\bullet) \circ \phi$$

which holds in particular with $k_{0,\iota}^\bullet$ and $k_{0,\gamma}^\bullet$ inserted for $g_{0,\iota}^\bullet$ and $g_{0,\gamma}^\bullet$, respectively.

We will moreover say that two isomorphic scaling limit theories have a *unique vacuum structure* if the connecting isomorphism also has the property

$$\omega_{0,\gamma} \circ \phi = \omega_{0,\iota}.$$

Following once more [10], one may now classify a given underlying QFTGA according to the following (mutually exclusive) possibilities:

- (1) All scaling limit QFTGAs are isomorphic, and $\mathfrak{F}_{0,\iota}$ is non-abelian. Then the underlying QFTGA is said to have a *unique quantum scaling limit*.
- (2) All scaling limit QFTGAs are isomorphic, and $\mathfrak{F}_{0,\iota}$ is abelian. In this case one says that the underlying QFTGA has a *classical scaling limit*.
- (3) There are scaling limit QFTGAs which are non-isomorphic. One then says that the underlying QFTGA has a *degenerate scaling limit*.

The interpretation of these cases is as in the case of observable algebras [10]; see this reference for further discussion. The first case would correspond to an underlying theory which has a single, stable ultraviolet fixed point. The second case is thought to correspond to an underlying theory which has no ultraviolet fixed point. The third case is in a sense intermediate, the underlying theory has a very irregular behaviour at small scales and has various, most likely unstable, ultraviolet fixed points.

We next put on record a result from [10] connecting the uniqueness of the scaling limit with the existence of a dilation symmetry in the scaling limit theories. The proof proceeds exactly as in the cited reference.

Proposition 2.6. [10] *Assume that all the scaling limit QFTGAs*

$$(\mathcal{F}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), U_{0,\iota}^\bullet(G_{0,\iota}^\bullet), \Omega_{0,\iota}, k_{0,\iota}^\bullet), \quad \iota \in \mathbb{I},$$

of the underlying QFTGA are isomorphic, i.e. that we are in case (1) or (2) of the just given classification. Then for each $\iota \in \mathbb{I}$ there exists a family $(\delta_\mu^{(0,\iota)})_{\mu>0}$ of automorphisms of $\mathfrak{F}_{0,\iota}$ acting as dilations in the corresponding scaling limit theory, which means that the following relations hold:

$$\begin{aligned}\delta_\mu^{(0,\iota)}(\pi_{0,\iota}(\mathfrak{F}(O))) &= \pi_{0,\iota}(\mathfrak{F}(\mu O)), \quad \mu > 0, \quad O \in \mathcal{K}, \\ \delta_\mu^{(0,\iota)} \circ \text{Ad } \mathcal{U}_{0,\iota}(a) &= \text{Ad } \mathcal{U}_{0,\iota}(\mu a) \circ \delta_\mu^{(0,\iota)}, \quad a \in \mathbb{R}^n, \quad \mu > 0, \\ \delta_\mu^{(0,\iota)} \circ \text{Ad } U_{0,\iota}(g) &= \text{Ad } U_{0,\iota}(g) \circ \delta_\mu^{(0,\iota)}, \quad g \in G, \quad \mu > 0.\end{aligned}$$

Furthermore, if the underlying QFTGA also has a unique vacuum structure in the scaling limit, then it follows that the family of dilations leaves the scaling limit states invariant: $\omega_{0,\iota} \circ \delta_\mu^{(0,\iota)} = \omega_{0,\iota}$, $\iota \in \mathbb{I}$, $\mu > 0$.

3 Scaling limits for QFTGAs with additional properties

In the present section we consider an underlying QFTGA with additional properties, such as Lorentz-covariance, spacelike clustering and geometric modular action, and we will investigate which further properties for the scaling limit theories ensue. More precisely, let $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$ be the underlying QFTGA, assumed to satisfy the conditions (QFTGA.1-5) of Def. 2.1. We will consider the following additional properties:

(QFTGA.6) (Lorentz covariance) There is a strongly continuous unitary representation $\tilde{\mathcal{L}}_+^\uparrow \ni L \mapsto \mathcal{W}(L) \in B(\mathcal{H})$ of the covering group of the proper, orthochronous Lorentz group \mathcal{L}_+^\uparrow (in d dimensions) on \mathcal{H} so that the following relations are fulfilled:

$$\begin{aligned}\tilde{\mathcal{W}}(L)\mathcal{W}(a) &= \mathcal{W}(\Lambda(L)a)\tilde{\mathcal{W}}(L), \\ \tilde{\mathcal{W}}(L)U(g) &= U(g)\tilde{\mathcal{W}}(L), \\ \tilde{\mathcal{W}}(L)\mathcal{F}(O)\tilde{\mathcal{W}}(L)^* &= \mathcal{F}(\Lambda(L)O), \quad \tilde{\mathcal{W}}(L)\Omega = \Omega\end{aligned}$$

for all $L \in \tilde{\mathcal{L}}_+^\uparrow$, $a \in \mathbb{R}^n$, $g \in G$ and $O \in \mathcal{K}$, where $\tilde{\mathcal{L}}_+^\uparrow \ni L \mapsto \Lambda(L) \in \mathcal{L}_+^\uparrow$ denotes the covering projection.

(QFTGA.7) (Irreducibility) $\mathfrak{F}' = \mathbb{C} \cdot 1$.

(QFTGA.8) (Spacelike clustering) We will assume that a uniform clustering bound holds on the vacuum (for spacetime dimension $d \geq 3$). To formulate this, we use the following notation. Elements in the $x^0 = 0$ hyperplane will be denoted by $\mathbf{x} \in \mathbb{R}^{n-1}$ and identified with $(0, \mathbf{x}) \in \mathbb{R}^n$. We define the derivation

$$\partial_0(F) := -i \left. \frac{d}{dx^0} \right|_{x^0=0} \alpha_{(x^0, \mathbf{0})}(F)$$

on the domain $D(\partial_0)$ of all $F \in \mathfrak{F}$ so that the (weak) derivative on the right hand side exists as an element in \mathfrak{F} . Note that $D(\partial_0)$ is a weakly dense subset of \mathfrak{F} . Then

our assumption on the existence of a uniform spacelike clustering bound is: There exists, for the given underlying QFTGA, a constant $c > 0$ so that for each double cone O_r having spherical base of radius r in the $x^0 = 0$ hyperplane there holds the bound

$$|\omega(F_1 \alpha_{\mathbf{x}}(F_2)) - \omega(F_1)\omega(F_2)| \leq \frac{cr^{n-1}}{|\mathbf{x}|^{n-2}} (||F_1|| ||\partial_0(F_2)|| + ||\partial_0(F_1)|| ||F_2||)$$

for all $F_1, F_2 \in \mathcal{F}(O_r) \cap D(\partial_0)$ as soon as $|\mathbf{x}| > 3r$.

(QFTGA.9) (Geometric modular action) A *wedge region* is any Poincaré-transformed copy of the so-called right wedge $W_R := \{(x^0, \dots, x^{n-1}) : |x^1| < x^0, x^0 > 0\}$. For this right wedge, we define the wedge-reflection map $r_R : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$r_R(x^0, x^1, x^2, \dots, x^{n-1}) := (-x^0, -x^1, x^2, \dots, x^{n-1}),$$

and the Lorentz-boosts

$$\begin{aligned} \Lambda_R(t)(x^0, x^1, x^2, \dots, x^{n-1}) \\ := (\cosh(t)x^0 + \sinh(t)x^1, \sinh(t)x^0 + \cosh(t)x^1, x^2, \dots, x^{n-1}). \end{aligned}$$

For any other wedge-region $W = LW_R$ with a suitable Poincaré-transformation L , we define $r_W := Lr_R L^{-1}$ and $\Lambda_W(t) := L\Lambda_R(t)L^{-1}$.

For each wedge region W in \mathbb{R}^n , the vacuum vector Ω of the underlying QFTGA is cyclic and separating for the von Neumann algebra $\mathcal{F}(W) = \{\mathcal{F}(O) : \overline{O} \subset W, O \in \mathcal{K}\}''$. Hence, there correspond to each wedge region W the Tomita-Takesaki modular objects J_W, Δ_W associated with $\mathcal{F}(W), \Omega$ [34]. It will then be assumed that, in the presence of (QFTGA.6), these modular objects act geometrically in the following way:

$$J_W \tilde{\mathcal{U}}(L) J_W = \tilde{\mathcal{U}}(\widetilde{\text{Adr}_W L}), \quad J_W \mathcal{U}(a) J_W = \mathcal{U}(r_W a), \quad L \in \tilde{\mathcal{L}}_+^\uparrow, \quad a \in \mathbb{R}^n, \quad (3.1)$$

$$\Delta_W^{it} = \tilde{\mathcal{U}}(\Lambda_W(2\pi t)), \quad t \in \mathbb{R}, \quad (3.2)$$

$$J_W \mathcal{F}(O) J_W = \mathcal{F}^t(r_W O), \quad O \in \mathcal{K}. \quad (3.3)$$

In these equations, we have denoted by $\widetilde{\text{Adr}_W}$ the lift of the adjoint action of r_W to $\tilde{\mathcal{L}}_+^\uparrow$, and by $\Lambda_W(t)$ the lift of $\Lambda_W(t)$ to $\tilde{\mathcal{L}}_+^\uparrow$ (both of which exist, cf. [17]). Moreover, we have introduced the so-called “twisted” local von Neumann algebras

$$\mathcal{F}^t(O) := V \mathcal{F}(O) V^*, \quad O \in \mathcal{K}, \quad (3.4)$$

where the twisting operator V is a unitary on \mathcal{H} defined by

$$V := (1 + i)^{-1} (1 + iU(k)). \quad (3.5)$$

Note that the algebras $\mathcal{F}(O_1)$ and $\mathcal{F}^t(O_2)$ commute for spacelike separated O_1 and O_2 on account of the assumed normal commutation relations.

We shall continue our investigation of the scaling limit theories of an underlying QFTGA satisfying some, or all, of the just stated additional conditions. In order to do that, we have to slightly re-define the scaling algebras $\underline{\mathfrak{F}}(O)$ when the underlying QFTGA satisfies Lorentz-covariance. For the remaining part of this article we adopt the following

Convention. Suppose that the underlying QFTGA satisfies also the condition of Lorentz-covariance (QFTGA.6). In this case, the local scaling algebras $\underline{\mathfrak{F}}(O)$, $O \in \mathcal{K}$, are defined as in Def. 2.2 but demanding in addition that the elements $\underline{F} \in \underline{\mathfrak{F}}(O)$ fulfill the also the condition

$$(e) \quad \|\underline{\alpha}_L(\underline{F}) - \underline{F}\| \rightarrow 0 \text{ as } L \rightarrow 1_{\tilde{\mathcal{L}}_+^\uparrow}$$

where

$$(\underline{\alpha}_L(\underline{F}))_\lambda := \mathcal{W}(L)\underline{F}_\lambda \mathcal{W}(L)^*.$$

Again, it is not difficult to demonstrate that, with that convention, the $\underline{\mathfrak{F}}(O)$ are C^* -algebras containing plenty of elements, and $\underline{\alpha}_{\mathbb{R}^n}$, $\underline{\beta}_G$ and $\underline{\alpha}_{\tilde{\mathcal{L}}_+^\uparrow}$ act as strongly continuous groups of automorphisms on $\underline{\mathfrak{F}}$ with the covariance properties (2.4) and, in addition,

$$\underline{\alpha}_L(\underline{\mathfrak{F}}(O)) = \underline{\mathfrak{F}}(\Lambda(L)O), \quad L \in \tilde{\mathcal{L}}_+^\uparrow, \quad O \in \mathcal{K}.$$

The following statement is again essentially a transcription of analogous results established for observable algebras in [10].

Proposition 3.1. *Suppose that the underlying QFTGA fulfills the conditions of Def. 2.2.*

1. *If the underlying QFTGA fulfills also Lorentz-covariance (QFTGA.6), then this property holds also for all scaling limit QFTGAs.*
2. *If the underlying QFTGA fulfills (QFTGA.6 & 7) and $n \geq 3$, then all scaling limit QFTGAs fulfill (QFTGA.6 & 7).*
3. *If the underlying QFTGA fulfills (QFTGA.8) and $n \geq 3$, then all scaling limit QFTGAs fulfill (QFTGA.7).*
4. *If the underlying QFTGA fulfills (QFTGA.6 & 9), then all scaling limit QFTGAs fulfill (QFTGA.6 & 9), too.*

Proof. Ad 1. This statement is proved in complete analogy to the corresponding statement in [10]; we note that for any scaling limit state $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega)$ (where ω is any locally normal state on $\underline{\mathfrak{F}}$) there holds $\omega_{0,\iota} \circ \underline{\alpha}_L = \omega_{0,\iota}$ and hence one obtains a unitary representation of $\tilde{\mathcal{L}}_+^\uparrow$ on $\mathcal{H}_{0,\iota}$ via setting

$$\tilde{\mathcal{W}}_{0,\iota}(L)\pi_{0,\iota}(\underline{F})\Omega_{0,\iota} := \pi_{0,\iota}(\underline{\alpha}_L(\underline{F}))\Omega_{0,\iota}, \quad L \in \tilde{\mathcal{L}}_+^\uparrow, \quad \underline{F} \in \underline{\mathfrak{F}}.$$

It is also easily checked that this unitary representation has all the properties analogous to those listed in (QFTGA.6) with respect to the scaling limit theory.

Ad 2. If the underlying theory has the additional properties (QFTGA.6 & 7), then this

entails that the underlying theory also has the property (QFTGA.8) according to a result by Araki, Hepp and Ruelle [3]; cf. also the proof of Lemma 4.3 in [10]. The statement then follows from 1. and 3.

Ad 3. Let $\underline{F}^{(1)}, \underline{F}^{(2)} \in \mathfrak{F}(O_{r'})$ and define, for some $h \in C_0^\infty(\mathbb{R}^n)$,

$$\underline{F}^{(j)} := \int d^n a h(a) \underline{\alpha}_a(\underline{F}^{(j)}), \quad j = 1, 2.$$

Then there is some $r > r'$ so that $\underline{F}^{(j)} \in \mathfrak{F}(O_r)$, and clearly $\underline{F}_\lambda^{(j)} \in \mathcal{F}(\lambda O) \cap D(\partial_0)$. We apply the uniform clustering bound to obtain, for each $\lambda > 0$ and $|\mathbf{x}| > 3r$,

$$\begin{aligned} & |\omega_\lambda(\underline{F}^{(1)} \underline{\alpha}_\mathbf{x}(\underline{F}^{(2)})) - \omega_\lambda(\underline{F}^{(1)}) \omega_\lambda(\underline{F}^{(2)})| \\ &= |\omega(\underline{F}_\lambda^{(1)} \alpha_{\lambda \mathbf{x}}(\underline{F}_\lambda^{(2)})) - \omega^\Omega(\underline{F}_\lambda^{(1)}) \omega(\underline{F}_\lambda^{(2)})| \\ &\leq \frac{c(\lambda r)^{n-1}}{|\lambda \mathbf{x}|^{n-2}} (\|\underline{F}_\lambda^{(1)}\| \|\partial_0(\underline{F}_\lambda^{(2)})\| + \|\partial_0(\underline{F}_\lambda^{(1)})\| \|\underline{F}_\lambda^{(2)}\|) \\ &\leq \frac{cr^{n-1}}{|\mathbf{x}|^{n-2}} (\|\underline{F}^{(1)}\| \|\partial_0(\underline{F}^{(2)})\| + \|\partial_0(\underline{F}^{(1)})\| \|\underline{F}^{(2)}\|), \end{aligned}$$

where we have defined $\partial_0(\underline{F}^{(j)}) := -i \frac{d}{dx^0} \big|_{x^0=0} \underline{\alpha}_{(x^0, \mathbf{0})}(\underline{F}^{(j)})$

and used the fact that

$\|\partial_0(\underline{F}_\lambda^{(j)})\| \leq \lambda^{-1} \|\partial_0(\underline{F}^{(j)})\|$. Now $\|\partial_0(\underline{F}^{(j)})\| < \infty$ by the definition of the $\underline{F}^{(j)}$, and taking the \limsup_λ on the left-hand side of the last inequality, one concludes that asymptotic spacelike clustering holds on the vacuum of each scaling limit theory since $\underline{F}^{(j)}$ approaches $\underline{F}'^{(j)}$ in the scaling algebra norm for $h \rightarrow \delta$. Because of normal commutation relations in each scaling limit QFTGA, this entails that $\mathfrak{F}'_{0,\iota} = \mathbb{C} \cdot 1$ holds in all scaling limit theories.

Ad 4. The proof proceeds analogously to the proof of Lemma 4.3 in [10]. \square

There is another result worth mentioning here which also generalizes a corresponding result established for observable algebras in [10] and connects a duality condition in scaling limit theories with the type of the local von Neumann algebras of the underlying QFTGA.

Theorem 3.2. *Suppose that the underlying QFTGA fulfills the assumptions of Def. 2.1. Moreover, suppose that there exists a scaling limit QFTGA*

$$(\mathcal{F}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), U_{0,\iota}^\bullet(G_{0,\iota}^\bullet), \Omega_{0,\iota}, k_{0,\iota}^\bullet)$$

having the property of “twisted wedge duality”,

$$\mathcal{F}_{0,\iota}(W)' = \mathcal{F}_{0,\iota}^t(r_W(W))$$

for some wedge region W in \mathbb{R}^d (with the definition of the twisted local von Neumann algebras analogous to (3.4) and (3.5) with respect to the corresponding objects in the scaling limit QFTGA); moreover, suppose that $\mathfrak{F}_{0,\iota} \neq \mathbb{C} \cdot 1$. In this case it holds that the local von Neumann algebras $\mathcal{F}(O)$ are of type III_1 for each double cone $O \subset W$ whose boundary intersects $\overline{W} \cap r_W(W)$, and for all translates of such double cones O . If twisted wedge duality holds for all wedge regions in some scaling limit QFTGA, then one concludes that $\mathcal{F}(O)$ is of type III_1 for all double cones.

We refer to Prop. 6.4 in [10] for a proof of this statement. We note also that according to the previous Proposition, the validity of conditions (QFTGA.6 & 7 & 9) in the underlying theory implies that the assumptions of Thm. 3.2 are fulfilled.

4 Quantum Field Systems with Gauge Symmetry

We now wish to investigate the scaling limits of QFTGAs that really correspond to superselection charges of a system of observables. Such QFTGAs are, more specifically, quantum field systems with gauge symmetry in the terminology of Doplicher and Roberts [15]. In order to summarize their definition here, and also for later reference, we first recapitulate some concepts of the Doplicher-Haag-Roberts approach to superselection theory, mainly from the sources [20, 30, 15].

This approach starts from the assumption that one is given an observable quantum system in a vacuum representation together with a further, distinguished set of representations modelling localized charges. The structure of an observable quantum system in a vacuum representation is described in terms of a collection of objects $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$ whose properties are assumed to be as follows.

- (a) \mathcal{A}_{vac} symbolizes a family $\{\mathcal{A}_{\text{vac}}(O)\}_{O \in \mathcal{K}}$ of von Neumann algebras in a separable Hilbert space \mathcal{H}_{vac} , subject to conditions of isotony (see above) and duality,

$$\mathcal{A}_{\text{vac}}(O)' = \mathcal{A}_{\text{vac}}(O') := \{\mathcal{A}_{\text{vac}}(O_1) : \overline{O_1} \subset O', O_1 \in \mathcal{K}\}'' ,$$

where O' denotes the open causal complement of O . Setting moreover $\mathfrak{A}_{\text{vac}} := \overline{\bigcup_O \mathcal{A}_{\text{vac}}(O)}^{C^*}$, it is assumed that $\mathfrak{A}'_{\text{vac}} = \mathbb{C} \cdot 1$.

- (b) $\mathcal{U}_{\text{vac}}(a)$, $a \in \mathbb{R}^n$, is a strongly continuous unitary representation of the translation group on \mathcal{H}_{vac} , acting covariantly on the family $\{\mathcal{A}_{\text{vac}}(O)\}_{O \in \mathcal{K}}$, and fulfilling the spectrum condition (see above). Furthermore, $\Omega_{\text{vac}} \in \mathcal{H}_{\text{vac}}$ is a unit vector which is left invariant by the action of $\mathcal{U}_{\text{vac}}(a)$, $a \in \mathbb{R}^n$.

Remark. Usually, also the assumption is made that the family $\{\mathcal{A}_{\text{vac}}(O)\}_{O \in \mathcal{K}}$ has the Borchers property (“Property B”). This property says that given $O, O_1 \in \mathcal{K}$ with $\overline{O} \subset O_1$ and a non-zero projection $E \in \mathcal{A}(O)$, then there is $V \in \mathcal{A}(O_1)$ with $VV^* = E$ and $V^*V = 1$. However, Roberts has shown [31] that this property can already be deduced from the other assumptions (essential being separability of \mathcal{H}_{vac} and the spectrum condition).

Given an observable quantum system $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$, one may look for representations of $\mathfrak{A}_{\text{vac}}$ describing the presence of charges. Following Doplicher, Haag and Roberts, one may consider the set $\mathfrak{P}^{\text{DHR}}$ of representations π of $\mathfrak{A}_{\text{vac}}$ which are unitarily equivalent to the vacuum representation in restriction to the causal complement of any double cone. That means, if $\mathfrak{A}_{\text{vac}}(O')$ is defined as the C^* -algebra generated by all $\mathcal{A}_{\text{vac}}(O_1)$ where $\overline{O_1} \subset O'$, then π is in $\mathfrak{P}^{\text{DHR}}$ if $\pi \upharpoonright \mathfrak{A}_{\text{vac}}(O')$ is unitarily equivalent to the identical representation of $\mathfrak{A}_{\text{vac}}(O')$ on $B(\mathcal{H}_{\text{vac}})$ for each $O \in \mathcal{K}$. Such representations describe superselection charges which are strictly localizable, see [20, 30] for further discussion.

We shall be interested only in the subset $\mathfrak{P}_{\text{cov}}^{\text{DHR}}$ of those π in $\mathfrak{P}^{\text{DHR}}$ which are translation-covariant, meaning that there is a strongly continuous representation $\mathcal{U}_\pi(a)$, $a \in \mathbb{R}^n$, of the translation group on the representation-Hilbertspace of π fulfilling the spectrum condition and the intertwining property

$$\text{Ad } \mathcal{U}_\pi(a)(\pi(A)) = \text{Ad } \mathcal{U}_{\text{vac}}(a)(A), \quad a \in \mathbb{R}^n, \quad A \in \mathfrak{A}_{\text{vac}}. \quad (4.1)$$

By identifying the representation-Hilbertspace \mathcal{H}_π with \mathcal{H}_{vac} , the set $\mathfrak{P}_{\text{cov}}^{\text{DHR}}$ may alternatively (and equivalently) be described in terms of the set Δ_t^{cov} of covariant, localized and transportable endomorphisms of $\mathfrak{A}_{\text{vac}}$. Here, an endomorphism $\rho : \mathfrak{A}_{\text{vac}} \rightarrow \mathfrak{A}_{\text{vac}}$ is called localized in $O \in \mathcal{K}$ if $\rho(A) = A$ holds for all $A \in \mathfrak{A}_{\text{vac}}(O')$. It is called transportable if, given an arbitrary region $O_1 \in \mathcal{K}$, there exists a unitary V so that $V\rho(\cdot)V^*$ is an endomorphism of \mathfrak{A} localized in O_1 ; one can show that V may be chosen as an element of \mathfrak{A} .

An element $\rho \in \Delta_t^{\text{cov}}$ is called *irreducible* if $\rho(\mathfrak{A}_{\text{vac}})' = \mathbb{C} \cdot 1$, and the set Sect^{cov} of all equivalence classes

$$[\rho] := \{V\rho(\cdot)V^* : V^* = V^{-1} \in \mathfrak{A}_{\text{vac}}\}$$

for irreducible $\rho \in \Delta_t^{\text{cov}}$ is called the set of *translation-covariant superselection sectors* of the given observable quantum system $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$.

If $\rho, \rho' \in \Delta_t^{\text{cov}}$, one defines by $\mathcal{I}(\rho, \rho')$ the set of *intertwiners* between ρ and ρ' as the set of all $T \in \mathfrak{A}_{\text{vac}}$ which satisfy

$$T\rho(A) = \rho'(A)T, \quad A \in \mathfrak{A}_{\text{vac}}.$$

Strictly speaking, one should refer to $\mathcal{I}(\rho, \rho')$ as the set of **global intertwiners** between ρ and ρ' . Given $O_1 \in \mathcal{K}$ and $\rho, \rho' \in \Delta_t^{\text{cov}}$ localized in O_1 , one can introduce $\mathcal{I}(\rho, \rho')_O$, the set of **local intertwiners** with respect to the localization region $O \supset O_1$, as consisting of all $T \in \mathfrak{A}_{\text{vac}}$ fulfilling

$$T\rho(A) = \rho'(A)T, \quad A \in \mathcal{A}_{\text{vac}}(O).$$

Hence it is obvious that $\mathcal{I}(\rho, \rho')_O \supset \mathcal{I}(\rho, \rho')$ for all $O \in \mathcal{K}$, and in Sec. 6 we will link the question if local and global intertwiners are equivalent, i.e. if $\mathcal{I}(\rho, \rho')_O = \mathcal{I}(\rho, \rho')$ holds for all $O \in \mathcal{K}$, to the preservice of charges in the scaling limit.

Presently, we need to very briefly summarize some further concepts of charge superselection theory (see, e.g. [30] for a more detailed account). First, one can introduce for $T_1 \in \mathcal{I}(\rho_1, \rho'_1)$ and $T_2 \in \mathcal{I}(\rho_2, \rho'_2)$ a product operation $T_1 \times T_2$ yielding an element in $\mathcal{I}(\rho_1\rho_2, \rho'_1\rho'_2)$. There is then a distinguished family of intertwiners $\epsilon(\rho_1, \rho_2) \in \mathcal{I}(\rho_1\rho_2, \rho_2\rho_1)$, for irreducible $\rho_1, \rho_2 \in \Delta_t^{\text{cov}}$, characterized by the property that it describes the exchange in the intertwiner product according to

$$(T_2 \times T_1)\epsilon(\rho_1, \rho_2) = \epsilon(\rho'_1, \rho'_2)(T_1 \times T_2), \quad T_j \in \mathcal{I}(\rho_j, \rho'_j),$$

together with the properties $\epsilon(\rho_1, \rho_2) = 1_{\rho_1\rho_2}$ if the localization regions of ρ_1 and ρ_2 are spacelike separated, and $\epsilon(\rho_2, \rho_1)\epsilon(\rho_1, \rho_2) = 1_{\rho_1\rho_2}$. Moreover, one can show that each irreducible $\rho \in \Delta_t^{\text{cov}}$ possesses a left inverse φ_ρ , i.e. a positive linear map on $\mathfrak{A}_{\text{vac}}$ which

preserves the unit and fulfills $\varphi_\rho(A\rho(B)) = \varphi_\rho(A)B$. Then there is for ρ a number λ_ρ so that

$$\varphi_\rho(\epsilon(\rho, \rho)) = \lambda_\rho 1.$$

The number λ_ρ depends only on the equivalence class $[\rho]$ of ρ and is called the *statistics parameter* of the corresponding superselection sector. If $\lambda_\rho \neq 0$, then the superselection sector is said to have *finite statistics*. We define by $\text{Sect}_{\text{fin}}^{\text{cov}}$ the set of all translation-covariant superselection sectors of the underlying observable quantum system which have finite statistics, and by $\Delta_{\text{fin}}^{\text{cov}}$ the set of all endomorphisms ρ with $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$.

Finally, we need to recollect the notion of a conjugate charge. One can show (cf. e.g. [30]) that for each $\rho \in \Delta_{\text{fin}}^{\text{cov}}$ localized in $O \in \mathcal{K}$ there is some $\bar{\rho} \in \Delta_{\text{fin}}^{\text{cov}}$, also localized in O , together with isometries R and \bar{R} in $\mathcal{A}(O)$ which intertwine the endomorphisms $\bar{\rho}\rho$ and $\rho\bar{\rho}$, respectively, with the identical endomorphism of $\mathfrak{A}_{\text{vac}}$, that is,

$$\bar{\rho}(\rho(A))R = RA \quad \text{and} \quad \rho(\bar{\rho}(A))\bar{R} = \bar{R}A, \quad A \in \mathfrak{A}_{\text{vac}}.$$

In this case, one calls $[\bar{\rho}]$ the conjugate superselection sector of $[\rho]$ or, synonymously, the conjugate charge of $[\rho]$.

Doplicher and Roberts [15] have shown that one can construct from $\Delta_{\text{fin}}^{\text{cov}}$ and the interwiners a system of local field algebras, acted upon by a faithful unitary representation of a compact group — called the gauge group — such that the local algebras of the initially given observable quantum system are embedded in the local field algebras as exactly containing the invariant elements under the gauge group action. In more precise terms, they have shown that one can associate with $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}, \Omega_{\text{vac}})$ a *quantum field system with gauge symmetry* (QFSGS), defined as follows:

Definition 4.1. $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$ is a QFSGS for $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}, \Omega_{\text{vac}})$ and $\Delta_{\text{fin}}^{\text{cov}}$ if the following conditions hold:

(QFSGS.1) $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$ is a QFTGA; the Hilbert space on which the von Neumann algebras $\mathcal{F}(O)$ of $\mathcal{F} = \{\mathcal{F}(O)\}_{O \in \mathcal{K}}$ act will be denoted by \mathcal{H} . Moreover, $\mathfrak{F}' = \mathbb{C}1$.

(QFSGS.2) There is a C^* -algebraic monomorphism

$$\pi : \mathfrak{A}_{\text{vac}} \rightarrow \mathfrak{F}$$

containing the vacuum representation (i.e., the identical representation of $\mathfrak{A}_{\text{vac}}$ on \mathcal{H}_{vac}) as a sub-representation, and such that $\pi(\mathcal{A}_{\text{vac}}(O))$ consists exactly of all $A \in \mathcal{F}(O)$ having the property $U(g)AU(g)^* = A$ for all $g \in G$. We will use the shorter notation

$$\mathcal{A}(O) := \pi(\mathcal{A}_{\text{vac}}(O)).$$

Moreover, the sub-Hilbert-space \mathcal{H}_0 of \mathcal{H} which is generated by all vectors $\mathcal{A}(O)\Omega$, as O ranges over the double cones, is cyclic for the algebras $\mathcal{F}(O)$.

(QFSGS.3) Let $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$ be a superselection sector. Then there exists a finite dimensional, irreducible, unitary representation

$$v_{[\rho]} = (v_{[\rho]ji})_{i,j=1}^d$$

of G (acting as a matrix representation for some suitable $d = d_{[\rho]}$) so that, for each $O \in \mathcal{K}$, there is a multiplet ψ_1, \dots, ψ_d of elements in $\mathcal{F}(O)$ having the following properties:

$$U(g)\psi_i U(g)^* = \sum_{j=1}^d \psi_j v_{[\rho]ji}(g), \quad (4.2)$$

$$\psi_i^* \psi_j = \delta_{ij} \mathbf{1}, \quad \sum_{j=1}^d \psi_j \psi_j^* = \mathbf{1}, \quad (4.3)$$

$$\pi \circ \rho_O(A) = \sum_{j=1}^d \psi_j \pi(A) \psi_j^*, \quad A \in \mathfrak{A}_{\text{vac}}, \quad (4.4)$$

for some representer ρ_O of $[\rho]$ localized in O .

These properties fix $v_{[\rho]}$ to within unitary equivalence.

(QFSGS.4) $\mathcal{F}(O)$ is generated by $\mathcal{A}(O)$ and all multiplets ψ_j , $j = 1, \dots, d_{[\rho]}$, with the properties (4.2), (4.3), (4.4), as $[\rho]$ ranges over all superselection sectors in $\text{Sect}_{\text{fin}}^{\text{cov}}$. For each finite-dimensional, irreducible, unitary representation v of G there is some superselection sector $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$ so that $v = v_{[\rho]}$ where $v_{[\rho]}$ has the properties of (QFSGS.3).

The conditions for a QFSGS associated with $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$ and $\Delta_{\text{fin}}^{\text{cov}}$ are given here in a form slightly different from the statement in [15]; however, the present formulation is convenient for our purposes.

It is plain that a QFSGS is a QFTGA fulfilling additional properties. Condition (QFSGS.4) states, in particular, that $\text{Sect}_{\text{fin}}^{\text{cov}}$ can be identified with the dual, \widehat{G} , of the gauge group G . The connection between field algebra and superselection sectors is essentially expressed through the multiplet operators ψ_1, \dots, ψ_d with the properties listed in (QFSGS.3). In fact, the occurrence of such “charge multiplets” associated with the superselection sector $[\rho]$ is equivalent to the presence of the corresponding charge in the QFSGS $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$. This will, basically, be our starting point for formulating criteria that express “preservation of a charge” in the scaling limit.

5 Preservance of Charges in the Scaling Limit

Let us now discuss the problem of characterizing “preservation of charges in the scaling limit” in greater detail. To this end, let $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$ be a QFSGS associated with $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$ and $\Delta_{\text{fin}}^{\text{cov}}$. Since $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$ is a QFTGA, we can form the corresponding scaling algebra $\underline{\mathfrak{F}}$ as in Sec. 2. We may then define

$$\underline{\mathfrak{A}}(O) = \{ \underline{A} \in \underline{\mathfrak{F}}(O) : \underline{A}_\lambda \in \mathcal{A}(\lambda O) \},$$

and it is not difficult to see that $\underline{\mathfrak{A}}(O)$ consists exactly of the $\underline{A} \in \underline{\mathfrak{F}}(O)$ so that

$$\underline{\beta}_g(\underline{A}) = \underline{A}$$

for all $g \in G$.

Now let $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega)$ be a scaling limit state on $\underline{\mathfrak{F}}$, and denote by

$$(\mathfrak{F}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), U_{0,\iota}^\bullet(G_{0,\iota}^\bullet), \Omega_{0,\iota}, k_{0,\iota}^\bullet)$$

the corresponding scaling limit QFTGA. Let us also denote by

$$\mathcal{A}_{0,\iota}(O) = \pi_{0,\iota}(\underline{\mathfrak{A}}(O)), \quad O \in \mathcal{K},$$

the von Neumann algebra formed by the scaling limits of the observables of the underlying QFSGS, and define by

$$\mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet} = \{ \mathbf{f} \in \mathcal{F}_{0,\iota}(O) : U_{0,\iota}^\bullet(g^\bullet) \mathbf{f} = \mathbf{f} U_{0,\iota}^\bullet(g^\bullet) \quad \forall g^\bullet \in G_{0,\iota}^\bullet \}$$

the fixed point algebra of the gauge group action in the scaling limit. With this notation, and recalling that $\mathcal{H}_{0,\iota} = \overline{\mathfrak{F}_{0,\iota} \Omega_{0,\iota}}$, we find:

Lemma 5.1. (i) $\mathcal{A}_{0,\iota}(O) = \mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet}$, $O \in \mathcal{K}$.

(ii) Suppose that $\Omega_{0,\iota}$ is the unique (up to a phase) unit vector in $\mathcal{H}_{0,\iota}$ which is invariant under $\mathcal{U}_{0,\iota}(\mathbb{R}^n)$ (equivalently, $\mathfrak{F}'_{0,\iota} = \mathbb{C} \cdot 1$). If $\mathfrak{A}_{0,\iota} = \overline{\bigcup_O \mathcal{A}_{0,\iota}(O)}^{C^*}$ is abelian, then $\mathfrak{F}_{0,\iota} = \mathbb{C} \cdot 1$ and hence, $\mathcal{H}_{0,\iota} = \mathbb{C} \Omega_{0,\iota}$.

Proof. (i) Clearly, one has $\mathcal{A}_{0,\iota}(O) \subset \mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet}$. To show that the reverse inclusion holds, let $\mathbf{f} \in \mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet}$. Denote by $m_{0,\iota}(\mathbf{h}) = \int_G d\mu(g) U_{0,\iota}(g) \mathbf{h} U_{0,\iota}(g)^*$, $\mathbf{h} \in \mathfrak{F}_{0,\iota}$, the mean over the action of G on $\mathfrak{F}_{0,\iota}$. We have $m_{0,\iota}(\mathbf{f}) = \mathbf{f}$. Let $\underline{F}^{(n)}$, $n \in \mathbb{N}$, be a sequence of elements in $\underline{\mathfrak{F}}(O)$ so that $w\text{-}\lim_{n \rightarrow \infty} \pi_{0,\iota}(\underline{F}^{(n)}) = \mathbf{f}$. Such a sequence exists because, by a Reeh-Schlieder argument, $\Omega_{0,\iota}$ is separating for $\mathcal{F}_{0,\iota}(O)$. Using this separating property of $\Omega_{0,\iota}$ once more, also $m_{0,\iota}(\pi_{0,\iota}(\underline{F}^{(n)}))$ approximates \mathbf{f} weakly. On the other hand,

$$m_{0,\iota}(\pi_{0,\iota}(\underline{F}^{(n)})) = \int_G d\mu(g) \pi_{0,\iota}(\underline{\beta}_g(\underline{F}^{(n)})) = \pi_{0,\iota}(\int_G d\mu(g) \underline{\beta}_g(\underline{F}^{(n)})),$$

where we made use of the continuity of $\underline{\beta}_G$ in norm on the scaling algebra to interchange representation and integration. Since $\int_G d\mu(g) \underline{\beta}_g(\underline{F}^{(n)})$ is contained in $\underline{\mathfrak{A}}(O)$, we see that \mathbf{f} is weakly approximated by elements in $\mathcal{A}_{0,\iota}(O)$ and hence is itself contained in $\mathcal{A}_{0,\iota}(O)$.

(ii) Under the given hypotheses, a result by Buchholz (Lemma 3.1 in [7]) shows that $\mathfrak{A}_{0,\iota} = \mathbb{C} \cdot 1$. Hence, the strongly continuous group $\beta_g^{(0,\iota)} = \text{Ad } U_{0,\iota}(g)$, $g \in G$, of automorphisms on $\mathfrak{F}_{0,\iota}$ acts ergodically, meaning that $\beta_g^{(0,\iota)}(\mathbf{f}) = \mathbf{f}$ for all $g \in G$ implies $\mathbf{f} \in \mathbb{C} \cdot 1$. Using Thm. 4.1 in [22], it follows that the unique ergodic state for $\beta_G^{(0,\iota)}$ on $\mathfrak{F}_{0,\iota}$ is a trace. The scaling limit vacuum $\langle \Omega_{0,\iota}, \cdot \Omega_{0,\iota} \rangle$ is a pure $\beta_G^{(0,\iota)}$ -invariant state on $\mathfrak{F}_{0,\iota}$ and hence is a trace. (Purity of this state holds since the space of translation-invariant vectors in $\mathcal{H}_{0,\iota}$ is one-dimensional.) This implies

$$\langle \Omega_{0,\iota}, \mathbf{f}^* \mathcal{U}_{0,\iota}(x) \mathbf{f} \Omega_{0,\iota} \rangle = \langle \Omega_{0,\iota}, \mathbf{f} \mathcal{U}_{0,\iota}(-x) \mathbf{f}^* \Omega_{0,\iota} \rangle$$

for each $\mathbf{f} \in \mathcal{F}_{0,\iota}(O)$, $O \in \mathcal{K}$, and all $x \in \mathbb{R}^n$. Arguing with spectrum condition and clustering (as a consequence of the assumption that every translation-invariant vector in $\mathcal{H}_{0,\iota}$ is a multiple of $\Omega_{0,\iota}$) in the same manner as in the proof of Lemma 3.1 in [7], one concludes that $\mathbf{f} \in \mathbb{C} \cdot 1$. Hence $\mathfrak{F}_{0,\iota} = \mathbb{C} \cdot 1$. \square

The Lemma shows that all charges of the underlying QFSGS disappear in a scaling limit theory once the scaling limit theory is known to be classical for the observables, provided the underlying theory satisfies very general conditions such as clustering (QFTGA.8) or (for $n \geq 3$) Lorentz-covariance (QFTGA.6).

At this point, we should emphasize the distinction between charges in the scaling limit QFTGA which are “scaling limits of charges of the underlying QFSGS”, and “charges arising as superselection sectors of the scaling limit theory”, as was first discussed by D. Buchholz [6]. Charges of the first mentioned type correspond to the situation that $G_{0,\iota}^\bullet$ is non-trivial and hence $U_{0,\iota}^\bullet(G_{0,\iota}^\bullet)$ acts non-trivially (and faithfully) on $\mathfrak{F}_{0,\iota}$. In this case, the action of $U_{0,\iota}^\bullet(G_{0,\iota}^\bullet)$ on the elements of $\mathfrak{F}_{0,\iota}$ may be seen as a short-distance remnant of the action of $U(G)$ on \mathfrak{F} so that, correspondingly, the members of the spectrum $\widehat{G}_{0,\iota}^\bullet$ of $G_{0,\iota}^\bullet$ may be viewed as representing short-distance remnants of the charges in \widehat{G} of the underlying QFSGS. It is important to note that, to some extent, these charges of the scaling limit theory have been present in the underlying QFSGS. We will discuss this case in more detail below.

The second type of charges in the scaling limit arises in a different way. One may consider the scaling limit theory (induced by $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega)$)

$$(\mathcal{A}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), \Omega_{0,\iota})$$

which is gained from the observables of the underlying QFSGS as a new observable quantum system in its own right (provided it fulfills the assumptions of irreducibility). Then one can assign a set of superselection sectors $\text{Sect}_{\text{fin}}^{\text{cov}} = \text{Sect}_{\text{fin}}^{\text{cov}}(\mathcal{A}_{0,\iota})$ to this observable quantum system, and by the Doplicher-Roberts reconstruction theorem⁵, we can now associate to these data a QFSGS, which we may denote by

$$(\mathcal{F}^{(0,\iota)}, \mathcal{U}^{(0,\iota)}(\mathbb{R}^n), U^{(0,\iota)}(G^{(0,\iota)}), \Omega^{(0,\iota)}, k^{(0,\iota)}).$$

Thus, this QFSGS contains the superselection charges which arise in the scaling limit theory of the *observables* of the underlying QFSGS. In general, it may occur that $\mathfrak{F}_{0,\iota}$ is properly contained in $\mathfrak{F}^{(0,\iota)}$ and that $G_{0,\iota}^\bullet$ is a factor group of $G^{(0,\iota)}$ by some non-trivial normal subgroup, so that the QFTGA associated with $\mathfrak{F}_{0,\iota}$ may be viewed as a proper subtheory (in the sense of [15]) of the QFSGS associated with $\mathfrak{F}^{(0,\iota)}$. Buchholz [6] proposed to consider such a case as a criterion for confinement, since it models the situation where charges appear as superselection sectors of the (observables’) scaling limit theory which do not arise as scaling limits of charges that occur as superselection sectors in the underlying QFSGS. We refer to [6, 8] for further discussion, and we note that examples for superselection charges of this second type have been constructed for the Schwinger model in two spacetime dimensions [8, 11].

⁵Provided that $(\mathcal{A}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), \Omega_{0,\iota})$ fulfills all conditions for an observable quantum system in vacuum representation. See our discussion after Prop. 5.6, and Prop. 5.7.

In the present work, we shall restrict attention solely to charges in the scaling limit QFTGAs of an underlying QFSGS of the first mentioned type, i.e. which arise as “scaling limits” of charges present in the underlying QFSGS. Having clarified this basic point, we must find criteria which express that a charge of the underlying QFSGS has a non-trivial scaling limit. There are some prefatory observations which may be helpful as a guideline. We have already seen that the gauge group $G_{0,\iota}^\bullet = G_{0,\iota}/N_{0,\iota}$ of a scaling limit QFTGA is a factor group of $G_{0,\iota}$ which is itself a copy of G , the gauge group of the underlying QFSGS. It may in general happen that the normal subgroup $N_{0,\iota}$ is non-trivial, and hence that $G_{0,\iota}^\bullet$ is “smaller” than G . In this situation, certainly not all the charges of the underlying QFSGS will have counterparts in the scaling limit QFTGA. Thus, we will in general be confronted with a situation which is in a sense complementary to that of $\mathfrak{F}_{0,\iota} \subset \mathfrak{F}^{(0,\iota)}$ mentioned just before and where, morally, the scaling limit QFTGA associated with $\mathfrak{F}_{0,\iota}$ corresponds to a subtheory of the underlying QFSGS, at least as far as the charge structure is concerned.⁶ However, since there is no inclusion of $\mathfrak{F}_{0,\iota}$ into \mathfrak{F} , we need to establish a correspondence between elements in $\mathfrak{F}_{0,\iota}$ and in \mathfrak{F} which allows to decide if charges present in the underlying QFSGS are also present in the scaling limit.

As we have mentioned above, the presence of a superselection charge in the underlying QFSGS manifests itself through the presence of charge multiplets $\psi_1, \dots, \psi_d \in \mathfrak{F}$ which transform under a finite dimensional, irreducible, unitary representation $v_{[\rho]}$ as described in (QFSGS.3). This will be the starting point for our criterion of charge preservice in the scaling limit. To fix ideas, let $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$ denote the underlying QFSGS, and let $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$ be one of its superselection sectors, and pick some arbitrary $O \in \mathcal{K}$. Then there is a finite-dimensional, irreducible, unitary representation $v_{[\rho]}$ of G and, for each $\lambda > 0$, a multiplet of elements $\psi_1(\lambda), \dots, \psi_d(\lambda)$ in $\mathcal{F}(\lambda O)$ having the properties of (QFSGS.3) with respect to the localization region λO . We will refer to any such multiplet family $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ as a **scaled multiplet** for $[\rho]$. The principal idea is now to view the functions $\lambda \mapsto \psi_j(\lambda)$ as “would-be” elements of $\mathfrak{F}(O)$ and to follow their fate as λ approaches 0. However, these functions won’t satisfy the “phase-space constraint” condition (c) of Def. 2.2 which is essential in order to interpret them as orbits of field algebra elements under (abstract) renormalization group transformations. Hence, if $\omega_{0,\iota}$ is a scaling limit state, in general one can’t form $\pi_{0,\iota}(\psi_j(\cdot))$ since $\psi_j(\cdot)$ won’t belong to the scaling algebra \mathfrak{F} . But one can still check if, in the scaling limit, scaled multiplets become close to elements of $\pi_{0,\iota}(\mathfrak{F})$ so that they can effectively be regarded as representing elements in the scaling limit von Neumann algebras $\mathcal{F}_{0,\iota}(O) = \pi_{0,\iota}(\mathfrak{F}(O))$. We will introduce some new terminology which gives this idea a more precise shape.

Definition 5.2. Let $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega)$ be a scaling limit state of the underlying QFSGS. Then we say that a family $\mathbf{F} = \{\mathbf{F}(\lambda)\}_{\lambda>0}$ fulfilling (i) $\mathbf{F}(\lambda) \in \mathfrak{F}(\lambda O_1)$ for some $O_1 \in \mathcal{K}$, (ii) $\sup_{\lambda>0} \|\mathbf{F}(\lambda)\| < \infty$, and (iii) $\|\beta_g(\mathbf{F}(\lambda)) - \mathbf{F}(\lambda)\| \rightarrow 0$ for $g \rightarrow 1_G$, is *asymptotically contained* in $\mathcal{F}_{0,\iota}(O)$ if the following holds:

For each given $\epsilon > 0$ there are elements \underline{F} and \underline{F}' in $\mathfrak{F}(O)$ such that

$$\limsup_{\kappa} \left(\|(\mathbf{F}(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})\Omega\| + \|(\mathbf{F}(\lambda_\kappa) - \underline{F}'_{\lambda_\kappa})^*\Omega\| \right) < \epsilon, \quad (5.1)$$

⁶The dynamics of the theories corresponding to $\mathfrak{F}_{0,\iota}$ and \mathfrak{F} are expected to be different and so the former can’t be a subtheory of the latter in the full sense of the definition.

where the net $\{\lambda_\kappa\}_{\kappa \in K}$ of positive numbers converges to 0, with $\omega_{0,\iota} = \lim_\kappa \underline{\omega}_{\lambda_\kappa}$ on \mathfrak{F} .

We will next collect some immediate consequences of this definition; this requires yet some further notation. Given a finite-dimensional Lie group X endowed with a Borel measure μ which is invariant under group transformations (for our purposes, $X = \mathbb{R}^n$ or $X = \widetilde{\mathcal{P}}_+^\uparrow$), we call a sequence of functions $\{h_\nu\}_{\nu \in \mathbb{N}}$ of class $L^1(X, \mu) \cap C_0^\infty(X)$ a δ -sequence if $\text{supp } h_{\nu+1} \subset \text{supp } h_\nu$, $\bigcap_\nu \text{supp } h_\nu = 1_X$, $\sup_\nu \|h_\nu\|_{L^1} < \infty$, and if $\int_X h_\nu \chi d\mu$ converges to $\chi(1_X)$ as $\nu \rightarrow \infty$ for all continuous functions χ on X . Here, 1_X is the group unit element; note that $1_{\mathbb{R}^n} = 0$.

Lemma 5.3. *Let $\omega_{0,\iota}$ be a scaling limit state of the underlying QFSGS, and suppose that $\mathbf{F} = \{\mathbf{F}(\lambda)\}_{\lambda>0}$ is a family of elements in \mathfrak{F} with the properties as in the previous definition. Then the following statements are equivalent:*

- (a) \mathbf{F} is asymptotically contained in $\mathcal{F}_{0,\iota}(O)$ for all $O \supset \overline{O_1}$,
- (b) In the scaling limit, \mathbf{F} is approached in the $*$ -strong topology by elements in $\pi_{0,\iota}(\mathfrak{F}(O))$ in the following sense: Whenever $O \supset \overline{O_1}$, $\epsilon > 0$ and finitely many $\underline{F}^{(1)}, \dots, \underline{F}^{(N)} \in \mathfrak{F}$ are given, then there is an $\underline{F} \in \mathfrak{F}(O)$ fulfilling $\|\underline{F}\| \leq \sup_\lambda \|\mathbf{F}(\lambda)\|$ and

$$\limsup_\kappa \left(\|(\mathbf{F}(\lambda_\kappa) - \underline{F}_{\lambda_\kappa}) \underline{F}_{\lambda_\kappa}^{(j)} \Omega\| + \|(\mathbf{F}(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})^* \underline{F}_{\lambda_\kappa}^{(j)} \Omega\| \right) < \epsilon, \quad j = 1, \dots, N,$$

where $\{\lambda_\kappa\}_{\kappa \in K}$ is as in the previous definition,

- (c) Given any δ -sequence $\{h_\nu\}$ on \mathbb{R}^n , there holds

$$\lim_{(\kappa, \nu)} \left(\|(\underline{\alpha}_{h_\nu} \mathbf{F})(\lambda_\kappa) - \mathbf{F}(\lambda_\kappa) \Omega\| + \|(\underline{\alpha}_{h_\nu} \mathbf{F})(\lambda_\kappa) - \mathbf{F}(\lambda_\kappa)^* \Omega\| \right) = 0, \quad (5.2)$$

where the limit is taken with respect to the partial ordering on $K \times \mathbb{N}$ given by $(\kappa, \nu) > (\kappa', \nu') :\Leftrightarrow \kappa > \kappa'$ and $\nu > \nu'$, and

$$(\underline{\alpha}_h \mathbf{F})(\lambda) = \int d^n x h(x) \alpha_{\lambda x}(\mathbf{F}(\lambda)), \quad \lambda > 0, \quad h \in L^1(\mathbb{R}^n).$$

(The latter integral is to be interpreted in the weak topology on \mathfrak{F} ; $\{\lambda_\kappa\}_{\kappa \in K}$ is as before).

Proof. (a) \Rightarrow (c). Let $\epsilon > 0$ be arbitrary. Then we must show that there exist $\kappa_\epsilon \in K$ and $\nu_\epsilon \in \mathbb{N}$ so that

$$\|(\underline{\alpha}_{h_\nu} \mathbf{F})(\lambda_\kappa) - \mathbf{F}(\lambda_\kappa) \Omega\| + \|(\underline{\alpha}_{h_\nu} \mathbf{F})(\lambda_\kappa) - \mathbf{F}(\lambda_\kappa)^* \Omega\| < \epsilon \quad (5.3)$$

holds for all $(\kappa, \nu) > (\kappa_\epsilon, \nu_\epsilon)$. Writing $(\underline{\alpha}_h \underline{F})_\lambda = \int d^n x h(x) \alpha_{\lambda x}(\underline{F}_\lambda)$, we consider the estimate

$$\begin{aligned} \|(\underline{\alpha}_{h_\nu} \mathbf{F})(\lambda_\kappa) - \mathbf{F}(\lambda_\kappa)^\sharp \Omega\| &\leq \|(\underline{\alpha}_{h_\nu} \mathbf{F})(\lambda_\kappa) - (\underline{\alpha}_{h_\nu} \underline{F})_{\lambda_\kappa}^\sharp \Omega\| + \|(\underline{\alpha}_{h_\nu} \underline{F})_{\lambda_\kappa} - \underline{F}_{\lambda_\kappa}\|^\sharp \Omega\| \\ &\quad + \|(\underline{F}_{\lambda_\kappa} - \mathbf{F}(\lambda_\kappa))^\sharp \Omega\|, \end{aligned} \quad (5.4)$$

where $(\dots)^\sharp$ stands for either (\dots) or $(\dots)^*$. Now we use the fact that, owing to the definition of asymptotic containment, one may choose $\underline{F}, \underline{F}'$ in such a way that there is some κ_ϵ with $(\sup_\nu \|h_\nu\|_{L^1} \|(\underline{F}(\lambda_\kappa) - \underline{F}_{\lambda_\kappa}^\sharp)^\sharp \Omega\| < \epsilon/6$ for all $\kappa > \kappa_\epsilon$, where $\underline{F}^\sharp = \underline{F}$ or $\underline{F}^\sharp = \underline{F}'$ according if $(\dots)^\sharp = (\dots)$ or $(\dots)^\sharp = (\dots)^*$. Denoting by \hat{h} the Fourier transform of h and by $P = (P_\nu)_{\nu=0}^{n-1}$ the selfadjoint generators of the unitary translation group of the underlying QFSGS, the first term on the right hand side of (5.4) is seen to equal

$$\|\hat{h}_\nu(P)(\underline{F}(\lambda_\kappa) - \underline{F}_{\lambda_\kappa}^\sharp)^\sharp \Omega\| \leq \sup_\nu \|h_\nu\|_{L^1} \|(\underline{F}(\lambda_\kappa) - \underline{F}_{\lambda_\kappa}^\sharp)^\sharp \Omega\| < \epsilon/6$$

for all $\kappa > \kappa_\epsilon$. The second term on the right hand side of (5.4) can be estimated by

$$\sup_\nu \|h_\nu\|_{L^1} \sup_{x \in \text{supp } h_\nu} \|\underline{\alpha}_x(\underline{F}^\sharp) - \underline{F}^\sharp\|$$

and using the continuity of \underline{F}^\sharp with respect to $\underline{\alpha}_x$, this quantity may be made smaller than $\epsilon/6$ for all ν smaller than some suitable ν_ϵ . Summing up, we obtain (5.3) for all $(\kappa, \nu) > (\kappa_\epsilon, \nu_\epsilon)$.

(c) \Rightarrow (b). It holds that $\lambda \mapsto \underline{\Phi}_\lambda = (\underline{\alpha}_{h_\nu} \underline{F})(\lambda)$ is contained in $\underline{\mathfrak{F}}(O_\times)$ where O_\times is any double cone containing $O_1 + \text{supp } h_\nu$. A standard Reeh-Schlieder argument shows that, if W is any wedge region in the causal complement of O_\times , then $\mathcal{F}_{0,\ell}(W)\Omega_{0,\ell}$ is dense in $\mathcal{H}_{0,\ell}$ hence $\mathcal{F}_{0,\ell}^t(W)\Omega_{0,\ell}$ is dense in $\mathcal{H}_{0,\ell}$. As a consequence, there is for given $\underline{F}^{(j)} \in \underline{\mathfrak{F}}$ and given $\eta > 0$ some $B^{(j)} \in \underline{\mathfrak{F}}(W)$ so that, if $V_{0,\ell}$ is the natural twist on $\mathcal{F}_{0,\ell}$,

$$\|(\pi_{0,\ell}(\underline{F}^{(j)}) - V_{0,\ell}\pi_{0,\ell}(\underline{B}^{(j)}))\Omega_{0,\ell}\| = \lim_\kappa \|(\underline{F}_{\lambda_\kappa}^{(j)} - V\underline{B}_{\lambda_\kappa}^{(j)})\Omega\| < \eta.$$

Thus, making first η and then ν^{-1} small enough, one can arrange that

$$\begin{aligned} & \limsup_\kappa \left(\|(\underline{\Phi}_{\lambda_\kappa} - \underline{F}(\lambda_\kappa))\underline{F}_{\lambda_\kappa}^{(j)}\Omega\| + \|(\underline{\Phi}_{\lambda_\kappa} - \underline{F}(\lambda_\kappa))^*\underline{F}_{\lambda_\kappa}^{(j)}\Omega\| \right) \\ & \leq \limsup_\kappa \left(\|(\underline{\Phi}_{\lambda_\kappa} - \underline{F}(\lambda_\kappa))V\underline{B}_{\lambda_\kappa}^{(j)}\Omega\| + \|(\underline{\Phi}_{\lambda_\kappa} - \underline{F}(\lambda_\kappa))^*V\underline{B}_{\lambda_\kappa}^{(j)}\Omega\| \right) + 4\eta \sup_\kappa \|\underline{F}(\lambda_\kappa)\| \\ & = \limsup_\kappa \left(\|\underline{B}_{\lambda_\kappa}^{(j)}V^*(\underline{\Phi}_{\lambda_\kappa} - \underline{F}(\lambda_\kappa))\Omega\| + \|\underline{B}_{\lambda_\kappa}^{(j)}V^*(\underline{\Phi}_{\lambda_\kappa} - \underline{F}(\lambda_\kappa))^*\Omega\| \right) + 4\eta \sup_\kappa \|\underline{F}(\lambda_\kappa)\| \\ & \leq \limsup_\kappa \|\underline{B}^{(j)}\| \left(\|(\underline{\Phi}_{\lambda_\kappa} - \underline{F}(\lambda_\kappa))\Omega\| + \|(\underline{\Phi}_{\lambda_\kappa} - \underline{F}(\lambda_\kappa))^*\Omega\| \right) + 4\eta \sup_\kappa \|\underline{F}(\lambda_\kappa)\| \end{aligned}$$

can be made smaller than any given $\epsilon > 0$; then, for a sufficiently large ν , $\underline{\Phi}$ can be taken as the \underline{F} required in (b). Note that in passing from the second line to the third we have used that $V\underline{B}_\lambda^{(j)}V^*$ commutes with $(\underline{\Phi}_\lambda - \underline{F}(\lambda))$ and its adjoint, because of the localization properties of the operators involved.

The implication (b) \Rightarrow (a) is obvious. \square

Remark. In view of statement (b) of the previous Lemma, one might refer to our notion of asymptotic containment more precisely as $*$ -strong asymptotic containment. It should then be obvious how to introduce, e.g., the notion of strong or weak asymptotic containment in $\mathcal{F}_{0,\ell}(O)$ for families $\underline{F} = \{\underline{F}(\lambda)\}_{\lambda>0}$ fulfilling the properties as in 5.2. One could also drop condition (iii) on \underline{F} in the definition of asymptotic containment, then having to

define in Lemma 5.3 $\underline{\alpha}_h \mathbf{F}$ differently, cf. (2.5).

After these preparations, we can now present our criterion for preservice of charges in the scaling limit.

Definition 5.4. Let $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega)$ be a scaling limit state of the underlying QFSGS, and let $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$ be a superselection sector. Then we say that the charge $[\rho]$ is *preserved in the scaling limit QFTGA* of $\omega_{0,\iota}$ if, for each $O_1 \in \mathcal{K}$, there is some scaled multiplet $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ for $[\rho]$ with $\psi_j(\lambda) \in \mathcal{F}(\lambda O_1)$ such that all families $\{\psi_j(\lambda)\}_{\lambda>0}$, $j = 1, \dots, d$, are asymptotically contained in $\mathcal{F}_{0,\iota}(O)$ if $O \supset \overline{O_1}$.

Let us briefly convince ourselves that each family $\{\psi_j(\lambda)\}_{\lambda>0}$ of a scaled multiplet satisfies the assumptions (i)–(iii) of Def. 5.2. Clearly, only condition (iii) need be checked, and one has

$$\sup_{\lambda} \|\beta_g(\psi_j(\lambda)) - \psi_j(\lambda)\| = \sup_{\lambda} \left\| \sum_{i=1}^d \psi_i(\lambda) (v_{[\rho]ij}(g) - \delta_{ij}) \right\| \leq d \max_{i,j} |v_{[\rho]ij}(g) - \delta_{ij}|$$

where the last term tends to 0 if $g \rightarrow 1_G$ if G is a continuous group.

We remark that, in view of part (c) of Lemma 5.3, a similar criterion has been used recently by Morsella [25]. Part (c) of Lemma 5.3 also provides some insight into the basic mechanism which might cause charges to disappear in the scaling limit. To elaborate on that, we consider a scaled multiplet $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ for the charge $[\rho]$. Moreover, for $h \in L^1(\mathbb{R}^n)$ with compact support and $h \geq 0$, $\int d^n x h(x) = 1$, we define

$$\underline{\Phi}_{\lambda}^{(h,j)} = (\underline{\alpha}_h \psi_j)(\lambda), \lambda > 0.$$

Now by Lemma 5.3 it follows that for the charge $[\rho]$ to be preserved in the scaling limit QFTGA of $\omega_{0,\iota}$, one must be able to choose a scaled multiplet and h in such a way that $\|\pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}\|$ comes arbitrarily close to 1. It could however happen that for all scaled multiplets and any choice of h one ends up with

$$\|\pi_{0,\iota}(\underline{\Phi}^{(h,j)*})\Omega_{0,\iota}\| = 0,$$

which also implies $\pi_{0,\iota}(\underline{\Phi}^{(h,j)}) = 0$ since $\Omega_{0,\iota}$ is separating for the local field algebras of the scaling limit QFTGA. We can interpret this as follows. The convolution of the scaled charge multiplets $\psi_j(\lambda)$ with respect to the scaled action of the translations, which produces elements $\underline{\Phi}^{(h,j)}$ in \mathfrak{F} , results in an energy damping of the charged states that are obtained by applying the $\underline{\Phi}_{\lambda}^{(h,j)*}$ to the vacuum vector Ω . This energy damping scales inversely, that is, proportional to λ^{-1} , to the localization scale of the $\underline{\Phi}_{\lambda}^{(h,j)}$. Depending on the dynamics of the underlying QFSGS, it may happen that the amount of energy-momentum required to create the charged vectors $\psi_j(\lambda)^* \Omega$ from the vacuum in a small region of scale λ is typically larger than $\sim \lambda^{-1}$, e.g. of the type $\sim \lambda^{-q}$ with some $q > 1$. In this case, the energy damping leads to a “blotting out” of the charged contributions of $\underline{\Phi}_{\lambda}^{(h,j)} \Omega$, resulting in the vanishing of the norm of these vectors as λ approaches 0.

In other words, our preservice criterion amounts essentially to requiring that the energy-momentum needed to localize the considered charges is only restricted by Heisenberg’s principle, and, also in view of the specific phase space properties of renormalization

group orbits encoded in the scaling algebra construction, it is evident that a condition of this kind is needed in order to single out “elementary, pointlike” charges, which survive the scaling limit (cf. also mechanism (B) below).

Let us sketch two — quite distinct — physical mechanisms that may account for the disappearance of charges in the scaling limit.

- (A) There is a strongly attractive force between the charges at short distances. This may have the effect that certain “compounds” of charges are dynamically more favourable than single charges. That is to say, it may cost far less energy to create a compound of several charges at small scales than the single charges contained in the compound. In this case, the compound charges could survive the scaling limit (i.e. be preserved), while certain single charges disappear since their creation costs too much energy at small scales. The compound charges preserved in the scaling limit could then well be invariant under some normal subgroup of the gauge group of the underlying quantum field theory. In a sense, this mechanism is complementary to that of confinement at finite distances of charges which would be viewed as “free” charges in the short-distance scaling limit (asymptotic freedom) as in QCD. There, one expects that the colour charges correspond to charges which are present as superselection charges of a scaling limit quantum field theory (corresponding to field multiplets in $\mathcal{F}^{(0,\iota)}$, not in $\mathcal{F}_{0,\iota}$), while in the underlying quantum field theory, at finite scale, only colour-neutral compounds of the colour-charges appear.
- (B) The charges are strongly repellent at short distances.⁷ In this situation, in order to localize two or more charges in a small spacetime region of scale λ , one requires more energy than of the order of λ^{-1} . This will typically lead to disappearance of certain charges in the scaling limit in the following way. If $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ is a scaled multiplet for some charge $[\rho]$, then there acts the tensor product representation $\bar{v}_{[\rho]} \otimes \bar{v}_{[\rho]}$ of the gauge group on the space of vectors spanned by $\psi_j^*(\lambda)\psi_k^*(\lambda)\Omega$, for each $\lambda > 0$. This tensor product representation can be decomposed into a sum of irreducibles, so that

$$\psi_j^*(\lambda)\psi_k^*(\lambda)\Omega = \sum_{\ell,m} a_{\ell,m}(\lambda)\psi_{\ell,m}^*(\lambda)\Omega$$

with suitable coefficients $a_{\ell,m}(\lambda)$, where the $\{\psi_{1,m}(\lambda), \dots, \psi_{d_m,m}(\lambda)\}_{\lambda>0}$ are scaled multiplets corresponding to the charges $[\rho_m]$ labelled by m which appear in the decomposition of $\bar{v}_{[\rho]} \otimes \bar{v}_{[\rho]}$. Assuming now that the interaction between charges of type $[\rho]$ is strongly repellent at short distances this will, in keeping with our discussion on the energy damping caused by applying $\underline{\alpha}_h$, typically result in

$$\pi_{0,\iota}(\underline{\alpha}_h(\psi_j^*\psi_k^*))\Omega_{0,\iota} = 0$$

for some indices j, k , and therefore in

$$\pi_{0,\iota}(\underline{\alpha}_h(\psi_{\ell,m}^*))\Omega_{0,\iota} = 0$$

for some indices ℓ, m .

⁷This mechanism has been pointed out to us by D. Buchholz

Concerning the question whether our criterion for preservice of charges is fulfilled in certain quantum field models, we note that the charges of the Majorana-Dirac field satisfy indeed this criterion in all scaling limit states (see Appendix A). We also remark that all charges in a dilation covariant theory complying with the hypotheses of [29] are preserved in all scaling limit QFTGAs: if for a multiplet $\psi_j \in \mathcal{F}(O)$, $j = 1, \dots, d$, associated to a given sector $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$, we define $\psi_j(\lambda) = D(\lambda)\psi_j D(\lambda)^{-1}$, $j = 1, \dots, d$, $D(\lambda)$ being the unitary implementation of dilations, we get a scaled multiplet $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ for $[\rho]$ such that $\psi_j(\cdot) \in \mathfrak{F}(O)$, as can be easily derived from the commutation relations between dilations and translations, and then in particular $\psi_j(\cdot)$ is asymptotically contained in every $\mathcal{F}_{0,\iota}(O)$. (This proves incidentally that a dilation covariant theory admits no classical scaling limit as soon as it has multiplets with dimension $d > 1$).

Our criterion of charge preservice not only bars the situation of charge disappearance, but it even implies that the limits of $\pi_{0,\iota}(\underline{\Phi}^{(h,j)})$, $j = 1, \dots, d$, as h tends to the δ -measure, yield charge multiplets corresponding to the charge $[\rho]$ with respect to their transformation behaviour under the scaling limit gauge group. This is the content of the following statement.

Proposition 5.5. *Suppose that the charge $[\rho]$ is preserved in the scaling limit QFTGA of $\omega_{0,\iota}$. Let $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ be a scaled multiplet for $[\rho]$ which is asymptotically contained in $\mathcal{F}_{0,\iota}(O)$, and let $\underline{\Phi}^{(h,j)}$ be defined as before with respect to the $\{\psi_j(\lambda)\}_{\lambda>0}$.*

Then for any δ -sequence $\{h_\nu\}$ on \mathbb{R}^n , the limit operators

$$\psi_j = s\text{-}\lim_{\nu \rightarrow \infty} \pi_{0,\iota}(\underline{\Phi}^{(h_\nu,j)}) \quad \text{and} \quad \psi_j^* = s\text{-}\lim_{\nu \rightarrow \infty} \pi_{0,\iota}(\underline{\Phi}^{(h_\nu,j)})^* \quad (5.5)$$

exist, are independent of the chosen δ -sequence and are contained in $\mathcal{F}_{0,\iota}(\hat{O})$ whenever $\hat{O} \supset \overline{O}$. Furthermore, ψ_1, \dots, ψ_d forms a multiplet transforming under the adjoint action of $U_{0,\iota}^\bullet(G_{0,\iota}^\bullet)$ according to the irreducible, unitary representation $v_{[\rho]}$. More precisely, denoting by $G \ni g \mapsto g^\bullet \in G_{0,\iota}^\bullet$ the quotient map, there is a finite-dimensional, irreducible, unitary representation $v_{[\rho]}^\bullet$ of $G_{0,\iota}^\bullet$ so that $v_{[\rho]}^\bullet(g^\bullet) = v_{[\rho]}(g)$ for all $g \in G$ and

$$U_{0,\iota}^\bullet(g^\bullet)\psi_j U_{0,\iota}^\bullet(g^\bullet)^* = \sum_{i=1}^d \psi_i v_{[\rho]ij}^\bullet(g^\bullet), \quad g^\bullet \in G_{0,\iota}^\bullet.$$

Proof. First we need to establish existence of the limit. Let $\{h_\nu\}$ and $\{\tilde{h}_{\tilde{\nu}}\}$ be δ -sequences on \mathbb{R}^n . Choose any $\epsilon > 0$. Then one can find $\nu_0 > 0$ so that

$$\begin{aligned} ||(\pi_{0,\iota}(\underline{\Phi}^{(h_\nu,j)}) - \pi_{0,\iota}(\underline{\Phi}^{(\tilde{h}_{\tilde{\nu}},j)})\Omega_{0,\iota}|| &= \lim_{\kappa} ||(\underline{\Phi}_{\lambda_\kappa}^{(h_\nu,j)} - \underline{\Phi}_{\lambda_\kappa}^{(\tilde{h}_{\tilde{\nu}},j)})\Omega|| \\ &\leq \limsup_{\kappa} \left(||(\underline{\Phi}_{\lambda_\kappa}^{(h_\nu,j)} - \psi_j(\lambda_\kappa))\Omega|| + ||(\underline{\Phi}_{\lambda_\kappa}^{(\tilde{h}_{\tilde{\nu}},j)} - \psi_j(\lambda_\kappa))\Omega|| \right) < \epsilon \end{aligned}$$

if $\nu, \tilde{\nu} > \nu_0$. This shows that $\pi_{0,\iota}(\underline{\Phi}^{(h_\nu,j)})\Omega_{0,\iota}$ is a Cauchy sequence in $\nu \rightarrow \infty$ and hence has a limit in $\mathcal{H}_{0,\iota}$; it shows also that the limit is independent of the chosen δ -sequence. Since $\Omega_{0,\iota}$ is separating for the local scaling limit field algebras and $||\underline{\Phi}^{(h_\nu,j)}||$ is bounded uniformly in ν , one can thus conclude that $\pi_{0,\iota}(\underline{\Phi}^{(h_\nu,j)})$ converges strongly to some ψ_j which is contained in $\mathcal{F}_{0,\iota}(\hat{O})$ if $\hat{O} \supset \overline{O}$. Similarly one argues that $\pi_{0,\iota}(\underline{\Phi}^{(h_\nu,j)})^*$ converges strongly to ψ_j^* .

Next we demonstrate $\psi_j^* \psi_k = \delta_{jk} 1$. To this end, we observe that for any $\underline{F} \in \underline{\mathfrak{F}}$ there holds the following chain of equations,

$$\begin{aligned}
& \langle \pi_{0,\iota}(\underline{F}) \Omega_{0,\iota}, (\psi_j^* \psi_k - \delta_{jk} 1) \Omega_{0,\iota} \rangle \\
&= \lim_{\nu \rightarrow \infty} \lim_{\kappa} [\langle \underline{\Phi}_{\lambda_\kappa}^{(h_\nu, j)} \underline{F}_{\lambda_\kappa} \Omega, \underline{\Phi}_{\lambda_\kappa}^{(h_\nu, k)} \Omega \rangle - \delta_{jk} \langle \underline{F}_{\lambda_\kappa} \Omega, \Omega \rangle] \\
&= \lim_{\nu \rightarrow \infty} \lim_{\kappa} [\langle \psi_j(\lambda_\kappa) \underline{F}_{\lambda_\kappa} \Omega, \psi_k(\lambda_\kappa) \Omega \rangle - \delta_{jk} \langle \underline{F}_{\lambda_\kappa} \Omega, \Omega \rangle \\
&\quad + \langle (\underline{\Phi}_{\lambda_\kappa}^{(h_\nu, j)} - \psi_j(\lambda_\kappa)) \underline{F}_{\lambda_\kappa} \Omega, \psi_k(\lambda_\kappa) \Omega \rangle \\
&\quad + \langle \underline{\Phi}_{\lambda_\kappa}^{(h_\nu, j)} \Omega, (\underline{\Phi}_{\lambda_\kappa}^{(h_\nu, k)} - \psi_k(\lambda_\kappa)) \Omega \rangle] .
\end{aligned}$$

The expression on the third to last line is equal to 0 since $\psi_j(\lambda)^* \psi_k(\lambda) = \delta_{jk} 1$ by assumption, and the limits of the expressions on the last two lines vanish by the argument having led to the conclusion (c) \Rightarrow (b) in the proof of Lemma 5.3. This proves $\psi_j^* \psi_k = \delta_{jk} 1$ by the separating property of $\Omega_{0,\iota}$ for the local field algebras in the scaling limit.

The proof of $\sum_{j=1}^d \psi_j \psi_j^* = 1$ is completely analogous.

For the last part of the statement, we observe that

$$U_{0,\iota}(g) \psi_j U_{0,\iota}(g)^* = \sum_{k=1}^d \psi_k v_{[\rho]kj}(g), \quad g \in G,$$

is simply a consequence of

$$\underline{\beta}_g(\underline{\Phi}^{(h,j)}) = \sum_{k=1}^d \underline{\Phi}^{(h,k)} v_{[\rho]kj}(g), \quad g \in G;$$

this, in turn, can be seen from $\underline{\Phi}^{(h,j)} = \underline{\alpha}_h \psi_j$ and the commutativity of $\underline{\beta}_g$ and $\underline{\alpha}_x$.

On the other hand, from the definition of $N_{0,\iota}$ one obtains

$$\psi_j \Omega_{0,\iota} = U_{0,\iota}(n) \psi_j \Omega_{0,\iota} = \sum_{k=1}^d \psi_k v_{[\rho]kj}(n) \Omega_{0,\iota}$$

for all $n \in N_{0,\iota}$, and multiplying by ψ_i^* from the left yields $\delta_{ij} \Omega_{0,\iota} = v_{[\rho]ij}(n) \Omega_{0,\iota}$ for all $n \in N_{0,\iota}$. This shows $v_{[\rho]}(n) = 1$ (the unit matrix) for all $n \in N_{0,\iota}$ and hence there is an irreducible, unitary representation $v_{[\rho]}^\bullet$ of $G_{0,\iota}^\bullet$ so that $v_{[\rho]}^\bullet(g^\bullet) = v_{[\rho]}(g)$ for all $g \in G$, proving the last part of the statement. \square

There is an obvious connection between the scaling limits of scaled multiplets for a charge $[\rho]$ and the scaling limits of endomorphisms induced by scaled multiplets in case that $[\rho]$ is preserved in a scaling limit state. While fairly immediate, we put the corresponding result on record here.

Proposition 5.6. *Let $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega)$ and let $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$ be a charge of the underlying QFSGS which is preserved in the scaling limit QFTGA of $\omega_{0,\iota}$. Moreover, let $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ be a scaled multiplet for $[\rho]$ asymptotically contained in $\mathcal{F}_{0,\iota}(O)$ and let, with respect to this scaled multiplet, ψ_1, \dots, ψ_d be defined as in (5.5).*

Then for each $\underline{A} \in \underline{\mathfrak{A}}$ the family $\{\rho(\underline{A})(\lambda)\}_{\lambda>0}$ defined by

$$\rho(\underline{A})(\lambda) = \sum_{j=1}^d \psi_j(\lambda) \underline{A}_\lambda \psi_j(\lambda)^*$$

is asymptotically contained in $\mathfrak{A}_{0,\iota}$. Furthermore, for each δ -sequence $\{h_\nu\}$ on \mathbb{R}^n there holds

$$s\text{-}\lim_{\nu \rightarrow \infty} \pi_{0,\iota}(\underline{\alpha}_{h_\nu} \rho(\underline{A})) = \sum_{j=1}^d \psi_j \pi_{0,\iota}(\underline{A}) \psi_j^*, \quad \underline{A} \in \underline{\mathfrak{A}}; \quad (5.6)$$

and $\boldsymbol{\rho}$ defined by

$$\boldsymbol{\rho}(\boldsymbol{a}) = \sum_{j=1}^d \psi_j \boldsymbol{a} \psi_j^*, \quad \boldsymbol{a} \in \mathfrak{A}_{0,\iota}, \quad (5.7)$$

is a localized, transportable, irreducible endomorphism of $\mathfrak{A}_{0,\iota}$ which is moreover covariant and has finite statistics.

Proof. The asymptotic containment in $\mathfrak{A}_{0,\iota}$ of $\{\rho(\underline{A})(\lambda)\}_{\lambda>0}$ is simply a consequence of the asymptotic containment of each $\{\psi_j(\lambda)\}_{\lambda>0}$ in $\mathcal{F}_{0,\iota}(O)$ and the fact that $\rho(\underline{A})(\lambda) \in \mathcal{A}(\lambda(O_1 \cap O_2))$ for $\underline{A} \in \underline{\mathfrak{A}}(O_2)$, with the conventional assumption that $\psi_j(\lambda) \in \mathcal{F}(\lambda O_1)$. Owing to (5.7) and the properties of a multiplet, $\boldsymbol{\rho}$ is clearly a localized, irreducible endomorphism of $\mathfrak{A}_{0,\iota}$. The transportability may be seen as follows. According to the definition of preserved charge, there is for any double cone O_\times different from O a scaled multiplet for $[\rho]$, $\{\tilde{\psi}_1(\lambda), \dots, \tilde{\psi}_d(\lambda)\}_{\lambda>0}$, which is asymptotically contained in $\mathcal{F}_{0,\iota}(O_\times)$. In the same way as the $\{\psi_j(\lambda)\}_{\lambda>0}$ lead to multiplet operators ψ_d in $\mathcal{F}_{0,\iota}(\hat{O})$ for all $\hat{O} \supset \overline{O}$, the $\{\tilde{\psi}_j(\lambda)\}_{\lambda>0}$ lead to multiplet operators $\tilde{\psi}_j$ contained in $\mathcal{F}_{0,\iota}(\hat{O}_\times)$ for all $\hat{O}_\times \supset \overline{O}_\times$. For the corresponding endomorphism $\tilde{\boldsymbol{\rho}}$ it then holds that $\boldsymbol{T} \tilde{\boldsymbol{\rho}}(\cdot) = \boldsymbol{\rho}(\cdot) \boldsymbol{T}$ with the unitary intertwiner $\boldsymbol{T} = \sum_{j=1}^d \psi_j \tilde{\psi}_j^*$. Now it is easy to see that the family $\{T(\lambda)\}_{\lambda>0}$ defined by $T(\lambda) = \sum_{j=1}^d \psi_j(\lambda) \tilde{\psi}_j^*$ is asymptotically contained in $\mathfrak{A}_{0,\iota}(O_*)$ for some double cone O_* , and by an argument by now familiar, $\boldsymbol{T} = s\text{-}\lim_{\nu \rightarrow \infty} \pi_{0,\iota}(\underline{\alpha}_{h_\nu} T)$ showing that \boldsymbol{T} is contained in $\mathcal{A}_{0,\iota}(\hat{O}_*)$ for $\hat{O}_* \supset \overline{O}_*$. Covariance follows from a general argument: Given a multiplet ψ_1, \dots, ψ_d , it holds that $\boldsymbol{\rho}(U \boldsymbol{a} U^*) = W \boldsymbol{\rho}(\boldsymbol{a}) W^*$ for each unitary U with $W = \sum_{j=1}^d \psi_j U \psi_j^*$ which is itself unitary. Moreover, if a continuous unitary group $a \mapsto U(a)$, $a \in \mathbb{R}^n$, fulfills the spectrum condition, then $a \mapsto W(a) = \sum_{j=1}^d \psi_j U(a) \psi_j^*$ is clearly also a continuous unitary group fulfilling the spectrum condition. That $\boldsymbol{\rho}$ has finite statistics follows from the finiteness of the dimension d of the multiplet. \square

Thus we have now seen that a preserved charge gives rise to localized, transportable endomorphisms of the observable quantum system in the scaling limit. However, it is not clear if the scaling limit theories $(\mathcal{A}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), \Omega_{0,\iota})$ gained from the observables of the underlying QFSGS satisfy all the technical properties assumed to hold for an observable quantum system in vacuum representation. Namely, the following two conditions can not be asserted for $(\mathcal{A}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), \Omega_{0,\iota})$ from the general assumptions made so far: (1) separability of the Hilbert-space $\mathcal{H}_{0,\iota}^{(0)} = \overline{\mathfrak{A}_{0,\iota} \Omega_{0,\iota}}$ and (2) Haag-duality. Without these two conditions, one isn't really in the situation where standard superselection theory applies,

and so it is not really clear if one can construct a QFSGS for $(\mathcal{A}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), \Omega_{0,\iota})$ that could be compared to the scaling limit QFTGA of the underlying theory.

Separability of the vacuum Hilbert-space of an observable quantum system is expected to hold quite generally for physically realistic theories, and moreover can be concluded for scaling limit theories from a decent behaviour of the energy-level density of the states of the underlying quantum field theory at short scales [7]. On the other hand, the difficulty with the possible failure of Haag-duality can be overcome by passing to the “dual net” to $\mathcal{A}_{0,\iota}$. This is the family $\mathcal{A}_{0,\iota}^d = \{\mathcal{A}_{0,\iota}^d(O)\}_{O \in \mathcal{K}}$ of von Neumann algebras defined by

$$\mathcal{A}_{0,\iota}^d(O) := \mathcal{A}_{0,\iota}(O')'$$

where O' denotes the causal complement of O and $\mathcal{A}_{0,\iota}(O')$ is the von Neumann algebra generated by the $\mathcal{A}_{0,\iota}(O_1)$ with $O_1 \subset O'$. Note that with this definition, since the family $\mathcal{A}_{0,\iota}$ satisfies the condition of locality, it holds that $\mathcal{A}_{0,\iota}(O) \subset \mathcal{A}_{0,\iota}^d(O)$. Quite obviously, $\mathcal{U}_{0,\iota}(\mathbb{R}^n)$ extends to a covariant action of the translations on $\mathcal{A}_{0,\iota}^d$ fulfilling the spectrum condition. Now it is known [28] that, if the family $\mathcal{A}_{0,\iota}$ fulfills the condition of geometric modular action analogous to condition (QFTGA.9), then the condition of Haag duality is fulfilled for the dual net $\mathcal{A}_{0,\iota}^d$. In turn, if the observable system $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$ of the underlying QFSGS is Lorentz covariant (QFTGA.6) and fulfills geometric modular action (QFTGA.9), then — observing the convention stated in Sec. 3 — also each scaling limit theory $(\mathcal{A}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), \Omega_{0,\iota})$ fulfills QFTGA.9 (cf. Prop. 3.1,4). Moreover, it was shown in Sec. 3.4 of [30] that any localized, transportable endomorphism of $\mathcal{A}_{0,\iota}$ extends to a localized, transportable endomorphism of $\mathcal{A}_{0,\iota}^d$ provided the latter fulfills the condition of Haag-duality. Therefore we have deduced the following result:

Proposition 5.7. *Let the dimension of spacetime n be 3 or higher. Assume that the observable quantum system of the underlying QFSGS fulfills Lorentz covariance (QFTGA.6) and geometric modular action (QFTGA.9) and, moreover, that the scaling limit Hilbert-space $\mathcal{H}_{0,\iota}^{(0)} = \overline{\mathfrak{A}_{0,\iota}\Omega_{0,\iota}}$ is separable. Then $(\mathcal{A}_{0,\iota}^d, \mathcal{U}_{0,\iota}(\mathbb{R}^n), \Omega_{0,\iota})$ is an observable quantum system in vacuum representation fulfilling the conditions (a) and (b) at the beginning of Sec. 4. If a superselection charge $[\rho]$ of the underlying QFSGS is preserved in this scaling limit, then the corresponding $\boldsymbol{\rho}$ defined in (5.7) extends to a localized, transportable endomorphism of $\mathfrak{A}_{0,\iota}^d$ which is covariant and has finite statistics.*

The last result presented in this section concerns the preservice of the conjugate charge of a preserved charge. To this end, let us assume for the remainder of this section that the underlying QFSGS fulfills also the condition of geometric modular action as formulated in (QFTGA.9) in Sec. 3. (We note that this can be deduced already if a similar form of geometric modular action is initially only assumed to hold for the underlying observable quantum system provided it fulfills some mild additional conditions. We refer to [17, 18, 23] for discussion of this issue.) In this case, let ψ_1, \dots, ψ_d be a multiplet for the charge $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$, with all ψ_j contained in $\mathcal{F}(O)$ for some $O \in \mathcal{K}$, and assume that W is a wedge region containing O . Let J_W denote the Tomita-Takesaki modular conjugation associated with $\mathcal{F}(W)$ and the vacuum vector Ω . Then one can take an arbitrary multiplet ψ'_1, \dots, ψ'_d for $[\rho]$ with all $\psi'_j \in \mathcal{F}(r_W O)$, and define a new multiplet of operators $\bar{\psi}_j \in \mathcal{F}(O)$, $j = 1, \dots, d$, by

$$\bar{\psi}_j = J_W V \psi'_j V^* J_W$$

where V is the “twist” operator defined in (3.5). It is easy to check that the $\bar{\psi}_j$ indeed form a multiplet, i.e. $\sum_{j=1}^d \bar{\psi}_j \bar{\psi}_j^* = 1$ and $\bar{\psi}_j^* \bar{\psi}_k = \delta_{jk} 1$; however, since J_W is antilinear, this multiplet transforms under the gauge group action according to the conjugate representation $\bar{v}_{[\rho]}$ of $v_{[\rho]}$,

$$U(g) \bar{\psi}_j U(g)^* = \sum_{i=1}^d \bar{\psi}_i \bar{v}_{[\rho]ij}(g), \quad g \in G,$$

if ψ_1, \dots, ψ_d transforms under the gauge group according to $v_{[\rho]}$. This indicates that $\bar{\psi}_1, \dots, \bar{\psi}_d$ is a multiplet of the conjugate sector $[\bar{\rho}]$ of $[\rho]$. Indeed, writing

$$\rho(A) = \sum_{j=1}^d \psi_j A \psi_j^*, \quad \bar{\rho}(A) = \sum_{j=1}^d \bar{\psi}_j A \bar{\psi}_j^*, \quad R = \frac{1}{\sqrt{d}} \sum_{j=1}^d \bar{\psi}_j \psi_j, \quad \bar{R} = \frac{1}{\sqrt{d}} \sum_{j=1}^d \psi_j \bar{\psi}_j,$$

one can easily check that R and \bar{R} are isometries in $\mathcal{A}(O)$ and moreover, there holds

$$\bar{\rho}(\rho(A))R = RA \quad \text{and} \quad \rho(\bar{\rho}(A))\bar{R} = \bar{R}A, \quad A \in \mathfrak{A}.$$

(This can actually also be deduced from a rather more general argument of [17].)

Equipped with these observations, we can now state the result.

Theorem 5.8. *Suppose that the underlying QFSGS fulfills the conditions of Poincaré covariance (QFTGA.6) and of geometric modular action (QFTGA.9), and let $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}}(\omega)$ be one of its scaling limit states. Then a charge $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$ is preserved in the scaling limit state $\omega_{0,\iota}$ if and only if also the conjugate charge $[\bar{\rho}]$ is preserved.*

Proof. Assume that $[\rho]$ is preserved in the scaling limit state $\omega_{0,\iota}$ and let $O \in \mathcal{K}$. Then for $O_1 \in \mathcal{K}$ with $\overline{O_1} \subset O$ there is a scaled multiplet $\{\psi'_1(\lambda), \dots, \psi'_d(\lambda)\}_{\lambda>0}$ for $[\rho]$, contained in $\mathcal{F}(\lambda r_W O_1)$ and asymptotically contained in $\mathcal{F}_{0,\iota}(r_W O_1)$. Let $\bar{\psi}_j$ be defined by

$$\bar{\psi}_j(\lambda) = J_W V \psi'_j(\lambda) V^* J_W$$

where V is the “twist” operator (cf. eq. (3.5)) and J_W is the modular conjugation associated with $\mathcal{F}(W)$ and the vacuum vector Ω . Then $\{\bar{\psi}_1(\lambda), \dots, \bar{\psi}_d(\lambda)\}_{\lambda>0}$ is a scaled multiplet for the conjugate charge $[\bar{\rho}]$ and each $\bar{\psi}_j(\lambda)$ is contained in $\mathcal{F}(\lambda O_1)$. Moreover, for compactly supported $h \in L^1(\mathbb{R}^n)$ it holds that

$$(\underline{\alpha}_h \bar{\psi}_j)(\lambda) = J_W V (\underline{\alpha}_{h \circ r_W} \psi'_j)(\lambda) V^* J_W$$

and this shows that the $\{\bar{\psi}_j(\lambda)\}_{\lambda>0}$ are asymptotically contained in $\mathcal{F}_{0,\iota}(O)$. \square

Remarks. (i) Note that under the conditions of Thm. 5.8 one also obtains asymptotic scaling limit versions of the isometries which intertwine ρ and $\bar{\rho}$. More precisely, suppose that a charge $[\rho]$ is preserved in the scaling limit state $\omega_{0,\iota}$, and let ψ_1, \dots, ψ_d be a corresponding multiplet contained in $\mathcal{F}_{0,\iota}(O)$ induced by a scaled multiplet $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$. As the previous Theorem shows, there is then a conjugate multiplet $\bar{\psi}_1, \dots, \bar{\psi}_d$ in $\mathcal{F}_{0,\iota}(O)$

induced by a scaled multiplet $\{\bar{\psi}_1(\lambda), \dots, \bar{\psi}_d(\lambda)\}_{\lambda>0}$, and it is straightforward to show that $\mathbf{R} = d^{-1/2} \sum_{j=1}^d \bar{\psi}_j \psi_j$ and $\bar{\mathbf{R}} = d^{-1/2} \sum_{j=1}^d \psi_j \bar{\psi}_j$ are given by

$$\mathbf{R} = s\text{-}\lim_{\nu \rightarrow \infty} \frac{1}{\sqrt{d}} \sum_{j=1}^d \pi_{0,\iota}(\alpha_{h_\nu} \bar{\psi}_j \psi_j) \quad \text{and} \quad \bar{\mathbf{R}} = s\text{-}\lim_{\nu \rightarrow \infty} \frac{1}{\sqrt{d}} \sum_{j=1}^d \pi_{0,\iota}(\alpha_{h_\nu} \psi_j \bar{\psi}_j)$$

where $\{h_\nu\}$ is any δ -sequence on \mathbb{R}^n . Using this, one deduces

$$\bar{\rho}(\rho(a))\mathbf{R} = \mathbf{R}a \quad \text{and} \quad \rho(\bar{\rho}(a))\bar{\mathbf{R}} = \bar{\mathbf{R}}a, \quad a \in \mathfrak{A}_{0,\iota},$$

where ρ and $\bar{\rho}$ relate to the ψ_j and $\bar{\psi}_j$, respectively, as in (5.7).

(ii) In addition, one obtains also the following: Let $J_{W0,\iota}$ and $V_{0,\iota}$ denote the analogous objects to J_W and V in the scaling limit theory of $\omega_{0,\iota}$, then a conjugate charge multiplet $\bar{\psi}_1, \dots, \bar{\psi}_d$ to ψ_1, \dots, ψ_d is obtained by

$$\bar{\psi}_j = J_{W0,\iota} V_{0,\iota} \psi'_j V_{0,\iota}^* J_{W0,\iota}$$

whenever ψ'_1, \dots, ψ'_d is a multiplet equivalent to ψ_1, \dots, ψ_d localized in $r_W O$.

6 On Equivalence of Local and Global Intertwiners

In the present section we will address the question of equivalence of local and global intertwiners of superselection sectors. We shall extend an argument of Roberts [29] who considered the setting of dilation covariant quantum field theories, showing that the preservation of all charges in some scaling limit theories⁸ is, together with the assumption that the local field algebras $\mathcal{F}(O)$ are factors, sufficient for the equivalence of local and global intertwiners. Our main technical result is stated in the following Lemma.

Lemma 6.1. *Let $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$ be a superselection sector of the underlying QFSGS, let $O \in \mathcal{K}$, and suppose that there are (i) a scaling limit state $\omega_{0,\iota} \in \text{SL}^{\mathcal{F}}(\omega)$, (ii) a scaled multiplet $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ for $[\rho]$ with $\psi_j(\lambda) \in \mathcal{F}(\lambda O)$, (iii) some compactly supported, non-negative $h \in L^1(\mathbb{R}^n)$, such that*

$$||\pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}|| > 0, \quad j = 1, \dots, d,$$

where $\underline{\Phi}^{(h,j)} = (\alpha_h \psi_j)(\lambda)$. Then for all unitaries $U \in \mathcal{A}(O)' \cap \mathcal{F}(O)$ and all multiplets $\tilde{\psi}_1, \dots, \tilde{\psi}_d \in \mathcal{F}(O)$ for $[\rho]$ ($O \in \mathcal{K}$) there holds

$$\begin{aligned} \omega(\tilde{\psi}_j^* U^* \tilde{\psi}_k U) &= \delta_{jk} & \text{if } \beta_k(\tilde{\psi}_i) &= \tilde{\psi}_i \\ \omega(\tilde{\psi}_j^* \beta_k(U^*) \tilde{\psi}_k U) &= \delta_{jk} & \text{if } \beta_k(\tilde{\psi}_i) &= -\tilde{\psi}_i \end{aligned} \tag{6.1}$$

⁸As already remarked in sec. 5, all charges are preserved in all scaling limits of dilation covariant theories.

Proof. We will treat explicitly the “even” case in (6.1), the “odd” case being completely analogous. First we note that $||\pi_{0,\ell}(\underline{\Phi}^{(h,j)})\Omega_{0,\ell}|| > 0$ for any of the $j = 1, \dots, d$ implies that the $\pi_{0,\ell}(\underline{\Phi}^{(h,j)})\Omega_{0,\ell}$, $j = 1, \dots, d$, are linearly independent. To see this, note that the contrary assumption of linear dependence implies that there is an invertible $d \times d$ matrix $(u_{j\ell})$ so that $\sum_j \pi_{0,\ell}(\underline{\Phi}^{(h,j)})\Omega_{0,\ell}u_{j\ell} = 0$ for some ℓ . But this implies

$$0 = U_{0,\ell}(g) \sum_j \pi_{0,\ell}(\underline{\Phi}^{(h,j)})\Omega_{0,\ell}u_{j\ell} = \sum_{j,k} \pi_{0,\ell}(\underline{\Phi}^{(h,k)})\Omega_{0,\ell}v_{[\rho]kj}(g)u_{j\ell}$$

for all $g \in G$ and hence, since $v_{[\rho]}$ is irreducible, $\pi_{0,\ell}(\underline{\Phi}^{(h,j)})\Omega_{0,\ell} = 0$ for all j .

We further observe that it constitutes no restriction of generality to prove the statement of the theorem only for $O \in \mathcal{K}$ which contain the origin $0 \in \mathbb{R}^n$ in their spacelike boundary (i.e. the origin is contained both in the boundary of O and in the boundary of its spacelike complement) since the underlying QFSGS is translation covariant. Thus we continue to prove the statement for an arbitrary O of this type.

We begin by noting that from our observation above, the $\pi_{0,\ell}(\underline{\Phi}^{(h,j)})\Omega_{0,\ell}$, $j = 1, \dots, d$, span a d -dimensional subspace of $\mathcal{H}_{0,\ell}$. Now let $W \supset O$ be a wedge region containing the origin in its spacelike boundary. Then let W' be the wedge which is the causal complement of W , and let W'_h be a copy of W' shifted by some suitable spacelike vector into the interior of W' such that W'_h lies in the causal complement of $O + \text{supp } h$. By a standard Reeh-Schlieder argument $\mathcal{F}_{0,\ell}(W'_h)\Omega_{0,\ell}$ is dense in $\mathcal{H}_{0,\ell}$ and hence, choosing some $\epsilon > 0$ arbitrarily, there will be some double cone $\hat{O} \subset W'_h$ and $\underline{F}^{(1)}, \dots, \underline{F}^{(d)} \in \underline{\mathfrak{F}}(\hat{O})$ such that

$$|\langle \pi_{0,\ell}(\underline{F}^{(j)})^*\Omega_{0,\ell}, \pi_{0,\ell}(\underline{\Phi}^{(h,k)})\Omega_{0,\ell} \rangle - \delta_{jk}| = |\omega_{0,\ell}(\underline{F}^{(j)}\underline{\Phi}^{(h,k)}) - \delta_{jk}| < \epsilon.$$

Now let $(\lambda_\kappa)_{\kappa \in K}$ be a subnet of the positive reals, converging to 0, with $\omega_{0,\ell} = \lim_{\kappa} \omega_{\lambda_\kappa}$ on $\underline{\mathfrak{F}}$. Since $\underline{F}_{\lambda_\kappa}^{(j)}\underline{\Phi}_{\lambda_\kappa}^{(h,k)}$ converges weakly to a multiple of 1 owing to $\bigcap_{O \ni 0} \mathcal{F}(O) = \mathbb{C}1$ (see [29]), we obtain

$$\omega_{0,\ell}(\underline{F}^{(j)}\underline{\Phi}_{\lambda_\kappa}^{(h,k)}) = \lim_{\kappa} \omega(\underline{F}_{\lambda_\kappa}^{(j)}\underline{\Phi}_{\lambda_\kappa}^{(h,k)}) = \lim_{\kappa} \omega'(\underline{F}_{\lambda_\kappa}^{(j)}\underline{\Phi}_{\lambda_\kappa}^{(h,k)})$$

for each locally normal state ω' on $\underline{\mathfrak{F}}$, and this implies

$$|\lim_{\kappa} \omega'(\underline{F}_{\lambda_\kappa}^{(j)}\underline{\Phi}_{\lambda_\kappa}^{(h,k)}) - \delta_{jk}| < \epsilon$$

whenever ω' is locally normal. On the other hand, since $||(\alpha_{\lambda x}(U) - U)\Omega|| \rightarrow 0$ as $\lambda \rightarrow 0$ uniformly for x ranging over compact sets, it follows that

$$\begin{aligned} & \omega(U^* \underline{F}_{\lambda_\kappa}^{(j)} \underline{\Phi}_{\lambda_\kappa}^{(h,k)} U) \\ &= \int d^n x h(x) \omega(\alpha_{\lambda_\kappa x}(U^*) \underline{F}_{\lambda_\kappa}^{(j)} \alpha_{\lambda_\kappa x}(\psi_k(\lambda_\kappa)) U) + o(\lambda_\kappa) \\ &= \int d^n x h(x) \omega(U^* \alpha_{-\lambda_\kappa x}(\underline{F}^{(j)})_{\lambda_\kappa} \psi_k(\lambda_\kappa) \alpha_{-\lambda_\kappa x}(U)) + o(\lambda_\kappa), \end{aligned}$$

where $o(\lambda)$ tends to 0 for $\lambda \rightarrow 0$, and we have used invariance of the vacuum state ω under the action of the translations α_x . We have also inserted the definition of the $\underline{\Phi}^{(h,k)}$, so that the scaled multiplets $\psi_k(\lambda)$ appear here.

Next we write $\psi_j(\lambda = 1) = \psi_j$, and we notice that $\psi_j(\lambda) = T_\lambda \psi_j$ where $T_\lambda = \sum_{j=1}^d \psi_j(\lambda) \psi_j^*$ is contained in $\mathcal{A}(O)$, and thus commutes with $U \in \mathcal{A}(O)' \cap \mathcal{F}(O)$. We note also that for every $B \in \mathfrak{F}$ we have,

$$\omega(U^* \alpha_{-\lambda_\kappa x}(\underline{F}_{\lambda_\kappa}^{(j)}) B) = \omega(\alpha_{-\lambda_\kappa x}(\underline{F}_{\lambda_\kappa}^{(j)}) U^* B)$$

for $\lambda_\kappa \leq 1$ and $x \in \text{supp } h$ since then $\alpha_{-\lambda_\kappa x}(\underline{F}_{\lambda_\kappa}^{(j)}) \in \mathcal{F}(W')$ and $U^* \in \mathcal{A}(O)' \cap \mathcal{F}(O) \subset \mathcal{F}(W)$. Hence we get for $\lambda_\kappa \leq 1$,

$$\begin{aligned} & \int d^n x h(x) \omega(U^* \alpha_{-\lambda_\kappa x}(\underline{F}_{\lambda_\kappa}^{(j)})_{\lambda_\kappa} \psi_k(\lambda_\kappa) \alpha_{-\lambda_\kappa x}(U)) \\ &= \int d^n x h(x) \omega(\alpha_{-\lambda_\kappa x}(\underline{F}_{\lambda_\kappa}^{(j)}) T_{\lambda_\kappa} (\sum_{i=1}^d \psi_i \psi_i^*) U^* \psi_k \alpha_{-\lambda_\kappa x}(U)) \\ &= \sum_{i=1}^d \int d^n x h(x) \omega(\alpha_{-\lambda_\kappa x}(\underline{F}_{\lambda_\kappa}^{(j)}) T_{\lambda_\kappa} \psi_i \alpha_{-\lambda_\kappa x}(\psi_i^* U^* \psi_k U)) + p(\lambda_\kappa) \\ &= \sum_{i=1}^d \omega(\underline{F}_{\lambda_\kappa}^{(j)} \underline{\Phi}_{\lambda_\kappa}^{(h,i)} \psi_i^* U^* \psi_k U) + p(\lambda_\kappa) \end{aligned}$$

with some function $p(\lambda)$ tending to 0 as $\lambda \rightarrow 0$, where we used that

$$\lim_{\lambda \rightarrow 0} \|(\psi_i^* U^* \psi_k \alpha_{-\lambda x}(U) - \alpha_{-\lambda x}(\psi_i^* U^* \psi_k U)) \Omega\| = 0$$

uniformly for x ranging over compact sets. Also we used the translational invariance of ω again. Summing up these findings we have for $\lambda_\kappa \leq 1$,

$$\omega(U^* \underline{F}_{\lambda_\kappa}^{(j)} \underline{\Phi}_{\lambda_\kappa}^{(h,k)} U) = \sum_{i=1}^d \omega(\underline{F}_{\lambda_\kappa}^{(j)} \underline{\Phi}_{\lambda_\kappa}^{(h,i)} \psi_i^* U^* \psi_k U) + o(\lambda_\kappa) + p(\lambda_\kappa).$$

Making now use of the fact that for all normal states ω' it holds that

$$\lim_{\kappa} |\omega'(\underline{F}_{\lambda_\kappa}^{(j)} \underline{\Phi}_{\lambda_\kappa}^{(h,i)}) - \delta_{ji}| < \epsilon,$$

the previous equation yields, upon taking the limit over κ ,

$$|\omega(\psi_j^* U^* \psi_k U) - \delta_{jk}| < (d+1)\epsilon.$$

Here $\epsilon > 0$ was arbitrary, and hence we conclude that

$$\omega(\psi_j^* U^* \psi_k U) = \delta_{jk}$$

holds for all unitary $U \in \mathcal{A}(O)' \cap \mathcal{F}(O)$ and the special multiplet $\psi_j = \psi_j(\lambda = 1)$. However, given any other multiplet $\tilde{\psi}_j$ in $\mathcal{F}(O)$ for the charge $[\rho]$, there is the unitary $T = \sum_{j=1}^d \tilde{\psi}_j \psi_j^*$ in $\mathcal{A}(O)$ so that $\tilde{\psi}_j = T \psi_j$, and thus we obtain, for each unitary $U \in \mathcal{A}(O)' \cap \mathcal{F}(O)$,

$$\omega(\tilde{\psi}_j^* U^* \tilde{\psi}_k U) = \omega(\psi_j^* T^* U^* T \psi_k U) = \omega(\psi_j^* U^* \psi_k U) = \delta_{jk}$$

since U and T commute. \square

Now we make use of the following result which has been proved in [29] (using also [14]): If, for some $O \in \mathcal{K}$, there holds (6.1) for all charge multiplets $\tilde{\psi}_j$ (of all superselection sectors) contained in $\mathcal{F}(O)$ and for all unitaries U contained in $\mathcal{A}(O)' \cap \mathcal{F}(O)$, then

$$\mathcal{A}(O)' \cap \mathcal{F}(O) = \mathcal{F}(O)' \cap \mathcal{F}(O).$$

If moreover the local field algebras of the underlying QFSGS are factors, i.e. if

$$\mathcal{F}(O) \cap \mathcal{F}(O)' = \mathbb{C}1, \quad O \in \mathcal{K}, \quad (6.2)$$

then equivalence of local and global intertwiners ensues: Given $[\rho]$ and $[\rho']$ in $\text{Sect}_{\text{fin}}^{\text{cov}}$ it holds that

$$\mathcal{I}(\rho, \rho')_O = \mathcal{I}(\rho, \rho') \quad \text{for } \rho, \rho' \text{ localized in } O. \quad (6.3)$$

(Cf. Sec. 4 for the definition of $\mathcal{I}(\rho, \rho')_O$ and $\mathcal{I}(\rho, \rho')$.)

Corollary 6.2. *Suppose that all local field algebras of the underlying QFSGS are factors, i.e. that (6.2) holds for all $O \in \mathcal{K}$. Moreover, suppose that for each charge $[\rho] \in \text{Sect}_{\text{fin}}^{\text{cov}}$ there is some scaling limit state $\omega_{0,\iota} \in \text{SL}^{\tilde{\mathfrak{f}}}(\omega)$ (which may depend on $[\rho]$) such that $[\rho]$ is preserved in that scaling limit state. Then in the underlying QFSGS there holds the equivalence of local and global intertwiners (6.3).*

The factorial property of the local field algebras has been checked in free field models. Assuming that this is a general feature of quantum field theories, the assertion of the Corollary shows that part of the charge superselection structure is determined entirely locally if all charges are preserved in suitable scaling limit states; in other words, if the charges are, in this (somewhat generalized) sense, ultraviolet stable. For further discussion as to how much else of the superselection structure may be determined locally, we refer to [29].

7 Scaling Algebras for Quantum Field Systems Localized in Spacelike Cones

Up to this point, we have considered quantum fields localizable in arbitrary bounded open regions, corresponding by the Doplicher-Roberts reconstruction theorem to superselection charges of the DHR-type. There are more general types superselection charges whose localization properties with respect to the vacuum representation of the observables are weaker. Before we enter into discussion of this fact, let us first introduce the relevant terminology.

As before, we identify n -dimensional Minkowski spacetime with \mathbb{R}^n . Following [15], the timelike hyperbolic submanifold $\mathcal{D} = \{s \in \mathbb{R}^n : \eta_{\mu\nu} s^\mu s^\nu = -1\}$ will be taken to represent all points at spacelike infinity of n -dimensional Minkowski spacetime since each $s \in \mathcal{D}$ represents a spacelike direction of unit Minkowskian length. Let a pair of points s_+, s_- in \mathcal{D} be given, where $s_+ \in (s_- + V_+)$, then we call $D(s_+, s_-) = (s_+ + V_-) \cap (s_- + V_+) \cap \mathcal{D}$

a **double cone at spacelike infinity** with future direction s_+ and past direction s_- . Then a **spacelike cone** is a set of the form

$$C = a + \{\lambda D(s_+, s_-) : \lambda > 0\}$$

where a is any element of \mathbb{R}^n and $D(s_+, s_-)$ is any double cone at spacelike infinity. A spacelike cone is thus a conic set extending to spacelike infinity having its apex at a ; it can be viewed as the set of points lying in a certain opening angle around a spacelike direction. With this definition, each spacelike cone is causally complete, i.e. taking its double causal complement reproduces each spacelike cone. For further properties of spacelike cones, we refer the reader to the discussion in the appendix of [15]. We will denote by \mathcal{S} the set of all spacelike cones.

A profound analysis by Buchholz and Fredenhagen [9] has shown that in a theory with no massless excitations a general superselection charge is generically localized in spacelike cones. Namely, they have proven that for any massive single particle representation π of \mathcal{A}_{vac} (i.e. π is a translation covariant representation having no translation invariant vector and the single particle states are separated from the continuum by a gap in the spectrum of the corresponding translations representation \mathcal{U}_π), there exists an irreducible vacuum representation $\tilde{\pi}_{\text{vac}}$ (i.e. a translation covariant representation with a translation invariant vector) such that, for any spacelike cone C , $\pi \upharpoonright \mathfrak{A}_{\text{vac}}(C')$ is unitarily equivalent to $\tilde{\pi}_{\text{vac}} \upharpoonright \mathfrak{A}_{\text{vac}}(C')$, $\mathfrak{A}_{\text{vac}}(C')$ being the C^* -algebra generated by all $\mathcal{A}_{\text{vac}}(O)$ with $\overline{O} \subset C'$.

Then, even in absence of massless particles, as in massive non-abelian gauge theories, DHR localization could be too strong a requirement. That such sectors really should arise in this kind of theories is suggested by the fact that spacelike cones can be thought as idealizations of flux tubes joining pairs of infinitely separated opposite gauge charges.

While in the case of strictly localizable field operators it was meaningful to define the scaling algebra of a QFTGA and to study the corresponding scaling limit theories even for the case that the QFTGA has not all the features of a QFSGS, the situation is somewhat different for the case of field operators which are only localizable in spacelike cones. To illustrate this, consider the case that one is given a collection of von Neumann algebras $\{\mathcal{F}(C)\}_{C \in \mathcal{S}}$ indexed by double cone (to be viewed as localization regions). Then one may fix, say, some spacelike cone C having its apex at the origin, and consider uniformly norm-bounded functions $\underline{F} : \mathbb{R}^+ \rightarrow \mathcal{F}(C)$. In order to take up the ideas that led to the definition of the scaling algebra for strictly localizable fields, however, one would now have to further restrict these functions in a manner expressing that \underline{F}_λ becomes localized near the origin as λ tends to 0. But since $\lambda C = C$ for all $\lambda > 0$, it is obvious that imposing $\underline{F}_\lambda \in \mathcal{F}(\lambda C)$ leads to no restriction in localization at all, and hence the localization constraint has to be implemented by making use of additional structure. And this can be achieved if it is assumed that the collection of von Neumann algebras $\{\mathcal{F}(C)\}_{C \in \mathcal{S}}$ belongs to a QFSGS corresponding to BF-type superselection charges, where one can exploit the strict localization properties of the quantum system of observables.

Therefore, let us now sketch, following [15], the Doplicher-Roberts reconstruction theorem for the case of charges of BF-type, which is very much in parallel to the discussion of the DHR case in Sec. 4. The starting point is again an observable quantum system $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$ fulfilling the properties listed at the beginning of Sec. 4. Then one may, as in [9], consider the set \mathfrak{P}^{BF} of representations which, upon restriction to the

spacelike complements of arbitrary spacelike cones, are unitarily equivalent to some fixed vacuum representation, which may be assumed to be the identical one. That means, in view of the above discussion, π is in \mathfrak{P}^{BF} iff $\pi \upharpoonright \mathfrak{A}_{\text{vac}}(C')$ is unitarily equivalent to the identical representation of $\mathfrak{A}_{\text{vac}}(C')$ on $B(\mathcal{H}_{\text{vac}})$. Then we restrict to the subset $\mathfrak{P}_{\text{cov}}^{\text{BF}}$ of such representations which are translation-covariant, where the definition of covariance is exactly as in Sec. 4 (cf. eqn. (4.1)). It can again be shown that the set $\mathfrak{P}_{\text{cov}}^{\text{BF}}$ is in one-to-one correspondence with the set of $\Delta_t^{\text{BF},\text{cov}}$ of covariant, localized, transportable morphisms of $\mathfrak{A}_{\text{vac}}$, now taking values in $B(\mathcal{H}_{\text{vac}})$, which are defined similarly as in Sec. 4 with the difference that these morphisms are no longer localized in double cones but in spacelike cones. The concepts of translation-covariant superselection sectors and of (global) intertwiners carry over literally from the situation of Sec. 4. This is likewise the case for the intertwiner product, the notion of statistics and of conjugate charge, keeping in mind that all morphisms are now localized in spacelike cones and that all properties referring to localization and spacelike separation must take this into account. This understood, one is led to defining the set of transportable, irreducible, covariant morphisms $\Delta_{\text{fin}}^{\text{BF},\text{cov}}$ of BF-type that have finite statistics, and their corresponding unitary equivalence classes collected in $\text{Sect}_{\text{fin}}^{\text{BF},\text{cov}}$ representing the superselection charges of BF-type of the theory.

Definition 7.1. One says that a collection of objects $(\mathcal{F}, \mathcal{U}(\mathbb{R}^n), U(G), \Omega, k)$ is a QFSGS associated with $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$ and $\Delta_{\text{fin}}^{\text{BF},\text{cov}}$ if the following holds:

- (*) \mathcal{F} denotes a family of von Neumann algebras (called field algebras) $\{\mathcal{F}(C)\}_{C \in \mathcal{S}}$ on a separable Hilbert-space indexed by spacelike cones. The family satisfies the analogues of conditions (QFTGA.1-5) when changing double cone localization regions of the field algebras to spacelike cones, and observing the following alterations: Since the set \mathcal{S} isn't directed with respect to inclusion, there is no counterpart to the quasilocal algebra \mathfrak{F} , and the assumption of cyclicity has to be altered to demanding that the space generated by $\mathcal{F}(C)\Omega$, where C ranges over all of \mathcal{S} , is dense in \mathcal{H} .
- (**) The analogue of (QFSGS.2) holds when replacing double cone localization regions by spacelike cones, more precisely, there is a C^* -algebraic monomorphism $\pi : \mathfrak{A}_{\text{vac}} \rightarrow B(\mathcal{H})$ so that

$$\mathcal{A}(O) := \pi(\mathcal{A}_{\text{vac}}(O)) \subset \mathcal{F}(C)$$

holds for all double cones O and all spacelike cones $C \supset O$. Moreover, π contains the vacuum representation of $\mathfrak{A}_{\text{vac}}$ on \mathcal{H}_{vac} as a sub-representation, and

$$\mathcal{A}(C) := \bigvee_{O \subset C} \mathcal{A}(O) = \mathcal{F}(C)^G.$$

Furthermore, if one denotes, for $D(s_+, s_-)$ any double cone at infinity, the C^* -algebra generated by all $\mathcal{F}(a + \mathbb{R}^+ D(s_+, s_-))$, $a \in \mathbb{R}^n$, by $\mathcal{F}[D(s_+, s_-)]$, then there holds

$$\pi(\mathfrak{A}_{\text{vac}})' \cap \mathcal{F}[D(s_+, s_-)] = \mathbb{C}\mathbf{1}.$$

- (*) The analogue of (QFSGS.3) holds, with the difference that the multiplets ψ_1, \dots, ψ_d are now elements of the $\mathcal{F}(C)$, and correspondingly one has to change ρ_O to ρ_C , a representer of $[\rho]$ localized in C , in (4.3).
- (**) The analogue of (QFSGS.4) holds upon replacing double cones O by spacelike cones C .

As Doplicher and Roberts [15] have shown, there is for each observable quantum system $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$ together with $\Delta_{\text{fin}}^{\text{BF, cov}}$ an associated QFSGS. We will now assume that we are given such a QFSGS corresponding to BF-type superselection charges, and construct a scaling algebra for it.

In order to do so, we have to introduce some notation. First of all, we recall that α_a stands for the adjoint action of $\mathcal{U}(a)$, and β_g stands for the adjoint action of $U(g)$. Then, let us denote by $B_\beta(\mathbb{R}^+, B(\mathcal{H}))$ the C^* -algebra of all bounded functions $F : \mathbb{R}^+ \rightarrow B(\mathcal{H})$ with C^* -norm given by

$$\|F\| := \sup_{\lambda > 0} \|F(\lambda)\|$$

and with pointwise defined algebraic operations, where the functions are also assumed to be continuous with respect to the standard lift of the gauge group action, meaning that

$$\|\underline{\beta}_g(F) - F\| = \sup_{\lambda > 0} \|\beta_g(F(\lambda)) - F(\lambda)\| \rightarrow 0 \quad \text{for } g \rightarrow 1_G.$$

Then we define $C_\alpha(\mathbb{R}^+, B(\mathcal{H}))$ as the C^* -subalgebra of $B_\beta(\mathbb{R}^+, B(\mathcal{H}))$ whose elements fulfill

$$\|\underline{\alpha}_a(F) - F\| = \sup_{\lambda > 0} \|\alpha_{\lambda a}(F(\lambda)) - F(\lambda)\| \rightarrow 0 \quad \text{for } a \rightarrow 0.$$

With this notation, we are ready to define scaling algebras for field operators localizable in spacelike cones.

Definition 7.2. Let O be a double cone and $C \supset O$ a spacelike cone. We define the scaling algebra $\underline{\mathfrak{A}}(O)$ as in Def. 2.2, i.e. consisting of all $\underline{A} \in C_\alpha(\mathbb{R}^+, B(\mathcal{H}))$ where $\underline{A}_\lambda \in \mathcal{A}(\lambda O)$, $\lambda > 0$. Then we denote by $\underline{\mathfrak{A}}(O)_1$ the subset of all elements of $\underline{\mathfrak{A}}(O)$ whose norm is bounded by unity, and we define:

- (I) $\underline{\mathfrak{F}}(C, O)$ is the C^* -subalgebra of all F in $B_\beta(\mathbb{R}^+, B(\mathcal{H}))$ having the properties

$$F(\lambda) \in \mathcal{F}(\lambda C) \quad \text{and} \quad \limsup_{\lambda \rightarrow 0} (\sup\{\|[F(\lambda), \underline{A}_\lambda]\| : \underline{A} \in \underline{\mathfrak{A}}(O')_1\}) = 0, \quad (7.1)$$

where $[A, B] = AB - BA$ denotes the commutator.

- (II) $\underline{\mathfrak{L}}(C, O) := \underline{\mathfrak{F}}(C, O) \cap C_\alpha(\mathbb{R}^+, B(\mathcal{H}))$.

Some remarks about this definition are in order. The second condition in 7.1 expresses the fact that the field operators $F(\lambda)$ that we are considering are *asymptotically localized*, as $\lambda \rightarrow 0$, in the double cone λO , in the sense that their effect on measurements performed in the spacelike complement of this bounded region vanishes in the limit. Through this requirement, then, we implement in this more general case the basic idea of the scaling

algebra approach. As a physical motivation for such a condition, we may note that a behaviour of this kind is expected to show up at least in nonabelian asymptotically free gauge theories: as remarked by Buchholz and Fredenhagen, the spacelike cone in which BF charges are localized has to be thought of as a fattened version of a gauge flux string between two opposite charges, one of which has been shifted at spacelike infinity, and it is then natural to expect that such string should become weaker and weaker at small scales if the theory is asymptotically free, leaving in the limit a compactly localizable charge.

We define the auxiliary C*-algebra $\underline{\mathfrak{F}}^\times$ as the C*-subalgebra of $C_\alpha(\mathbb{R}^+, B(\mathcal{H}))$ generated by all the algebras $\underline{\mathfrak{F}}(C, O)$, and we note that for this system of algebras we have the obvious covariance properties

$$\underline{\alpha}_a(\underline{\mathfrak{F}}(C, O)) = \underline{\mathfrak{F}}(C + a, O + a), \quad \underline{\beta}_g(\underline{\mathfrak{F}}(C, O)) = \underline{\mathfrak{F}}(C, O),$$

for the actions $\underline{\alpha}_{\mathbb{R}^n}$ and $\underline{\beta}_G$ of the translations and gauge group defined above, so that these restrict to automorphic actions on $\underline{\mathfrak{F}}^\times$, denoted by the same symbols. For each normal state ω' on $B(\mathcal{H})$ we define the family of states $(\underline{\omega}'_\lambda)_{\lambda>0}$ on $\underline{\mathfrak{F}}^\times$ in analogy to the case of localized fields,

$$\underline{\omega}'_\lambda(\underline{F}) := \omega'(\underline{F}_\lambda), \quad \underline{F} \in \underline{\mathfrak{F}}^\times,$$

and $\text{SL}^{\mathfrak{F}^\times}(\omega') = \{\omega'_{0,\iota} : \iota \in \mathbb{I}\}$ will be the set of weak* limit points of the net $(\underline{\omega}'_\lambda)_{\lambda>0}$ (this is non-void, as in the localizable case), and will be called the set of *scaling limit states* of ω' .

Lemma 7.3. *Assume that the net of observable algebras in the vacuum representation satisfies the following condition: for each double cone O containing the origin, there holds*

$$\mathcal{A}_{\text{vac}}(O) = \bigvee_{O_0 \ni 0} \mathcal{A}_{\text{vac}}(O \cap O'_0), \quad (7.2)$$

where O_0 runs through all double cones containing the origin. Then $\text{SL}^{\mathfrak{F}^\times}(\omega')$ is independent of the normal state ω' .

Remark. The above condition (7.2) is suggested by the fact that $\bigcap_{O_0 \ni 0} \mathcal{A}_{\text{vac}}(O_0) = \mathbb{C}\mathbf{1}$, by Haag duality and by the time-slice axiom. Its validity can also be proven in free field models.

Proof of Lemma 7.3. As the union of all C*-algebras $\underline{\mathfrak{F}}(C, O)$ is norm dense in $\underline{\mathfrak{F}}^\times$, it is sufficient to show that, for any two normal states ω^1, ω^2 on $B(\mathcal{H})$, and for any choice of C, O with $O \subset C$ and any $\underline{F} \in \underline{\mathfrak{F}}(C, O)$, there holds

$$\lim_{\lambda \rightarrow 0} \underline{\omega}^1_\lambda(\underline{F}) - \underline{\omega}^2_\lambda(\underline{F}) = 0.$$

To this end, we adapt Roberts' argument [29] and assume that this is not true. Then we can find a subnet $(\underline{F}_{\lambda_\nu})_\nu$ of $(\underline{F}_\lambda)_{\lambda>0}$ weakly convergent to some $F_0 \in B(\mathcal{H})$ and such that

$$|\omega^1(F_0) - \omega^2(F_0)| = \lim_\nu |\underline{\omega}^1_{\lambda_\nu}(\underline{F}) - \underline{\omega}^2_{\lambda_\nu}(\underline{F})| > 0.$$

Now we intend to show that F_0 is a multiple of $\mathbf{1}$ which leads to a contradiction, and hence shows validity of the statement of the lemma. We first observe that, if D is the double

cone at spacelike infinity defined by the spacelike cone C , then from $\underline{F}_{\lambda_\nu} \in \mathcal{F}(\lambda_\nu C) \subset \mathcal{F}[D]$ for each ν , $F_0 \in \mathcal{F}[D]$ follows. Let then O_1 be a double cone containing the origin, take $A \in \mathfrak{A}(O'_1)$ such that $\|A\| \leq 1$ and $x \mapsto \alpha_x(A)$ is norm continuous, and define for each $\mu > 0$ an element $\underline{A}^{(\mu)} \in \mathfrak{A}(\frac{1}{\mu}O'_1)$ by

$$\underline{A}_\lambda^{(\mu)} := \begin{cases} A & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

Then since $\underline{A}^{(\mu)} \in \mathfrak{A}(O')_1$ for μ sufficiently small, from the asymptotic localizability of $\underline{F} \in \mathfrak{F}(C, O)$ it follows that

$$\limsup_{\lambda \rightarrow 0} \|\underline{F}_\lambda, A\| = \limsup_{\lambda \rightarrow 0} \|\underline{F}_\lambda, \underline{A}_\lambda^{(\lambda)}\| \leq \lim_{\lambda \rightarrow 0} \sup_{\underline{A} \in \mathfrak{A}(O')_1} \|\underline{F}_\lambda, \underline{A}_\lambda\| = 0,$$

which implies $[F_0, A] = 0$, so that, as the multiples of the A 's satisfying the stated requirements form a weakly dense set in each algebra $\mathcal{A}(O'_0) := \mathfrak{A}(O'_0)''$ with $O_0 \supset \overline{O}_1$, we get

$$F_0 \in \left[\bigvee_{O_0 \ni 0} \mathcal{A}(O'_0) \right]'$$

But from the assumption (7.2), together with local normality of the representation π , we have $\pi(\mathfrak{A}_{\text{vac}}) \subseteq \bigvee_{O_0 \ni 0} \mathcal{A}(O'_0)$, so that, by $(*)_{**}$ of Definition 7.1, $F_0 \in \pi(\mathfrak{A}_{\text{vac}})' \cap \mathcal{F}[D] = \mathbb{C}\mathbf{1}$. \square

In view of the above lemma, from now on we will only consider the scaling limit states $\text{SL}^{\mathfrak{F}^\times}(\omega)$, with $\omega := \langle \Omega, (\cdot)\Omega \rangle$ the underlying vacuum state. For $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}^\times}(\omega)$, let $(\pi_{0,\iota}^\times, \mathcal{H}_{0,\iota}^\times, \Omega_{0,\iota})$ be the associated GNS representation and let $\mathcal{U}_{0,\iota}^\times(a)$, $a \in \mathbb{R}^n$, $U_{0,\iota}^\times(g)$, $g \in G$, be respectively the translations group and gauge group representations, obtained as in the case of localized fields (Part 2 of Proposition 2.4), thanks to $\underline{\alpha}_{\mathbb{R}^n}$ - and $\underline{\beta}_G$ -invariance of $\omega_{0,\iota}$. We define then for each double cone O the von Neumann algebra

$$\mathcal{F}_{0,\iota}^\times(O) := \bigcap_{C \supset O} \pi_{0,\iota}^\times(\mathfrak{F}(C, O))'', \quad (7.3)$$

and correspondingly a cyclic Hilbertspace $\mathcal{H}_{0,\iota} := \overline{\bigcup_O \mathcal{F}_{0,\iota}^\times(O)\Omega_{0,\iota}}$, a net of von Neumann algebras over it given by $\mathcal{F}_{0,\iota}(O) := \mathcal{F}_{0,\iota}^\times(O) \upharpoonright \mathcal{H}_{0,\iota}$, a translation group representation $\mathcal{U}_{0,\iota}(a) := \mathcal{U}_{0,\iota}^\times(a) \upharpoonright \mathcal{H}_{0,\iota}$, $a \in \mathbb{R}^n$, and a gauge group representation $U_{0,\iota}^\bullet(g^\bullet) := U_{0,\iota}^\times(g) \upharpoonright \mathcal{H}_{0,\iota}$, $g^\bullet \in G_{0,\iota}^\bullet$, where, in analogy to the localizable case, $G_{0,\iota}^\bullet := G/N_{0,\iota}$ with $N_{0,\iota}$ the closed normal subgroup of G of the elements $g \in G$ such that $U_{0,\iota}^\times(g) \upharpoonright \mathcal{H}_{0,\iota} = \mathbf{1}_{\mathcal{H}_{0,\iota}}$, and $g \in G \mapsto g^\bullet \in G_{0,\iota}^\bullet$ the quotient map, so that $U_{0,\iota}$ is a faithful unitary representation of $G_{0,\iota}^\bullet$ on $\mathcal{H}_{0,\iota}$. We also denote by $\pi_{0,\iota}$ the subrepresentation of $\pi_{0,\iota}^\times$ determined by $\mathcal{H}_{0,\iota}$.

Proposition 7.4. *The quintuple $(\mathcal{F}_{0,\iota}, \mathcal{U}_{0,\iota}(\mathbb{R}^n), U_{0,\iota}(G_{0,\iota}^\bullet), \Omega_{0,\iota}, k_{0,\iota}^\bullet)$ defined above is a normal, covariant quantum field theory with gauge group action, which will be called a scaling limit QFTGA, corresponding to $\omega_{0,\iota}$, of the QFSGS determined by $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$ and $\Delta_{\text{fin}}^{\text{BF}, \text{cov}}$.*

Proof. The proof is completely analogous to the one of Proposition 2.4, so we don't repeat it here. The only thing that deserves a comment is the normality of commutation relations within $\mathcal{F}_{0,\iota}$. We first note that they hold for the system of algebras $\mathfrak{F}(C, O)$, in the sense that, by defining $\underline{F}_\pm := \frac{1}{2}(\underline{F} \pm \beta_k(\underline{F}))$, relations analogous to (2.1) are satisfied for $\underline{F}_i \in \mathfrak{F}(C_i, O_i)$, with spacelike separated C_i , $i = 1, 2$. Clearly this also carries over to the system of algebras $\pi_{0,\iota}^\times(\mathfrak{F}(C, O))''$, with respect to the grading defined by $\beta_k^{(0,\iota)}$. If then $\mathbf{f}_i \in \mathcal{F}_{0,\iota}^\times(O_i)$ with spacelike separated O_i , $i = 1, 2$, we can assume that $\mathbf{f}_i \in \pi_{0,\iota}^\times(\underline{F}(C_i, O_i))''$ with $C_i \supset O_i$, $i = 1, 2$, spacelike separated to each other, so that normal commutation relations also hold for the net $\mathcal{F}_{0,\iota}^\times$, and hence for $\mathcal{F}_{0,\iota}$. \square

The classification of the underlying theory in terms of the resulting structure of the scaling limit theories given in Section 2 can be clearly applied also here, the isomorphism notion being again the one of Definition 2.5.

We can also show a result analogous to Proposition 2.6, stating that if all the scaling limit QFTGAs are isomorphic, then they are dilation covariant. Since the formulation of this result and its proof are straightforward, we omit them.

It is also straightforward to show that if the underlying QFSGS is also Lorentz covariant, where Lorentz covariance is defined as in (QFTGA.6), understanding that spacelike cones substitute double cones,⁹ then the same is true for each scaling limit QFTGA, provided that in this case one considers the scaling algebras $\mathfrak{F}(C, O)$ obtained by redefining the C*-algebra $C_\alpha(\mathbb{R}^+, B(\mathcal{H}))$ appearing in Definition 7.2 as the C*-subalgebra of $B_\beta(\mathbb{R}^+, B(\mathcal{H}))$ whose elements \mathbf{F} fulfill

$$\|\underline{\alpha}_s(\mathbf{F}) - \mathbf{F}\| = \sup_{\lambda > 0} \|\alpha_{s_\lambda}(\mathbf{F}(\lambda)) - \mathbf{F}(\lambda)\| \rightarrow 0 \quad \text{for } s \rightarrow 1_{\tilde{\mathcal{P}}_+^\uparrow},$$

where $s := (L, a) \in \tilde{\mathcal{P}}_+^\uparrow$ is a generic element of the covering of the (proper orthochronous) Poincaré group $\tilde{\mathcal{P}}_+^\uparrow = \tilde{\mathcal{L}}_+^\uparrow \ltimes \mathbb{R}^n$, $s_\lambda := (L, \lambda a)$, and $\alpha_s := \text{Ad}\mathcal{U}(s)$, $\mathcal{U}(L, a) := \mathcal{U}(a)\tilde{\mathcal{U}}(L)$ (slightly abusing notation). We will denote by $\mathcal{U}_{0,\iota}(s)$, $s \in \tilde{\mathcal{P}}_+^\uparrow$, the corresponding unitary representation of the Poincaré group on $\mathcal{H}_{0,\iota}$, with respect to which $\mathcal{F}_{0,\iota}$ is covariant. We also note, for future use, that $\mathcal{U}_{0,\iota}(s) = \mathcal{U}_{0,\iota}^\times(s) \upharpoonright \mathcal{H}_{0,\iota}$, where $\mathcal{U}_{0,\iota}^\times(s)$ is the unitary representation of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}_{0,\iota}^\times$ defined by $\mathcal{U}_{0,\iota}^\times(s)\pi_{0,\iota}^\times(\underline{F})\Omega_{0,\iota} = \pi_{0,\iota}^\times(\alpha_s(\underline{F}))\Omega_{0,\iota}$.

We introduce here the standard notations

$$\alpha_s^{(0,\iota)\times} := \text{Ad}\mathcal{U}_{0,\iota}^\times(s), \quad \alpha_s^{(0,\iota)} := \text{Ad}\mathcal{U}_{0,\iota}(s), \quad s \in \tilde{\mathcal{P}}_+^\uparrow,$$

and, for simplicity, we will identify $L \in \tilde{\mathcal{L}}_+^\uparrow$ and $a \in \mathbb{R}^n$ with $(L, 0), (1_{\tilde{\mathcal{L}}_+^\uparrow}, a) \in \tilde{\mathcal{P}}_+^\uparrow$ respectively.

We close this section by putting on record a result generalizing Part 4 of Proposition 3.1.

Proposition 7.5. *Assume that the underlying QFSGS determined by BF sectors is Poincaré covariant and satisfies the condition of geometric modular action (QFTGA.9), where in this case $\mathcal{F}(W) := \bigvee_{\overline{C} \subset W} \mathcal{F}(C)$, and where equation (3.3) is substituted by*

$$J_W \mathcal{F}(C) J_W = \mathcal{F}^t(r_W C), \quad C \in \mathcal{S}. \quad (7.4)$$

⁹This happens for instance if one considers the QFSGS determined through the Doplicher-Roberts reconstruction theorem from the set of Poincaré covariant BF sectors.

Then each scaling limit QFTGA satisfies the condition (QFTGA.9) of geometric modular action.

Proof. Let $\tilde{\mathfrak{F}}_{0,\iota}^\times(W)$ be the C^* -algebra generated by all $\pi_{0,\iota}^\times(\mathfrak{F}(C, O))$ with $O \subset C$ and $\overline{C} \subset W$, and let $\mathfrak{F}_{0,\iota}^\times(W) := \tilde{\mathfrak{F}}_{0,\iota}^\times(W)''$. In Appendix B we prove that the scaling limit vacuum $\Omega_{0,\iota}$ is cyclic and separating for $\tilde{\mathfrak{F}}_{0,\iota}^\times(W)$ and that $\omega_{0,\iota}$ is a (-2π) -KMS state for the C^* -dynamical system $(\tilde{\mathfrak{F}}_{0,\iota}^\times(W), \alpha_{\Lambda_W(\cdot)}^{(0,\iota)\times})$, and then also for the W^* -dynamical system $(\mathfrak{F}_{0,\iota}^\times(W), \alpha_{\Lambda_W(\cdot)}^{(0,\iota)\times})$. Then, arguing as in the proof of Lemma 6.2 in [10] and denoting by $(\Delta_{0,\iota}^\times, J_{0,\iota}^\times)$ the modular objects determined by $(\tilde{\mathfrak{F}}_{0,\iota}^\times(W), \Omega_{0,\iota})$, it is easy to verify the relations

$$J_{0,\iota}^\times \mathcal{U}_{0,\iota}^\times(L, a) J_{0,\iota}^\times = \mathcal{U}_{0,\iota}^\times(\widetilde{\text{Ad}_{r_W} L}, r_W a), \quad (7.5)$$

$$(\Delta_{0,\iota}^\times)^{it} = \mathcal{U}_{0,\iota}^\times(\widetilde{\Lambda_W(2\pi t)}), \quad (7.6)$$

$$J_{0,\iota}^\times \pi_{0,\iota}^\times(\mathfrak{F}(C, O))'' J_{0,\iota}^\times = \pi_{0,\iota}^\times(\mathfrak{F}(r_W C, r_W O))''. \quad (7.7)$$

But since $\mathfrak{F}_{0,\iota}^\times(W) := \bigvee_{\overline{O} \subset W} \mathfrak{F}_{0,\iota}^\times(O)$ is contained in $\tilde{\mathfrak{F}}_{0,\iota}^\times(W)$, and $\mathfrak{F}_{0,\iota}(W) = \mathfrak{F}_{0,\iota}^\times(W) \upharpoonright \mathcal{H}_{0,\iota}$, it follows easily that $\omega_{0,\iota}$ is also (-2π) -KMS for $(\mathfrak{F}_{0,\iota}(W), \alpha_{\Lambda_W(\cdot)}^{(0,\iota)})$, so that we also get

$$\Delta_{0,\iota}^{it} = \mathcal{U}_{0,\iota}(\widetilde{\Lambda_W(2\pi t)}). \quad (7.8)$$

This, together with (7.6) and standard arguments of Tomita-Takesaki theory, implies that if $\mathbf{f} \in \mathfrak{F}_{0,\iota}(W)$ is analytic for $\alpha_{\Lambda_W(\cdot)}^{(0,\iota)}$ there holds

$$J_{0,\iota} \mathbf{f} \Omega_{0,\iota} = \mathcal{U}_{0,\iota}(\widetilde{\Lambda_W(-i\pi)}) \mathbf{f}^* \Omega_{0,\iota} = \mathcal{U}_{0,\iota}^\times(\widetilde{\Lambda_W(-i\pi)}) \mathbf{f}^* \Omega_{0,\iota} = J_{0,\iota}^\times \mathbf{f} \Omega_{0,\iota},$$

so that one obtains $J_{0,\iota}^\times \upharpoonright \mathcal{H}_{0,\iota} = J_{0,\iota}$ (since the analytic elements for $\alpha_{\Lambda_W(\cdot)}^{(0,\iota)}$ weakly dense in $\mathfrak{F}_{0,\iota}(W)$). The equations (7.5), (7.7) then imply geometric modular action for the scaling limit QFTGA. \square

8 Preservance of BF-type Charges in the Scaling Limit

We will now generalize the notion of preservance of charges given in Section 5 so as to encompass the more general situation of BF-type charges considered in the previous section. In particular, in view of the physical picture of asymptotically free theories discussed after Definition 7.2, and of the ensuing construction of the scaling algebras and scaling limit, we will formulate a criterion implying that a given BF charge of the underlying theory gives rise to a localizable charge in the scaling limit.

We consider then a Poincaré covariant observable net $(\mathcal{A}_{\text{vac}}, \mathcal{U}_{\text{vac}}(\tilde{\mathcal{P}}_+^\dagger), \Omega_{\text{vac}})$ and the corresponding Poincaré covariant QFSGS $(\mathcal{F}, \mathcal{U}(\tilde{\mathcal{P}}_+^\dagger), U(G), \Omega, k)$ determined by the set $\Delta_{\text{fin}}^{\text{BF}, \text{cov}}$ of finite statistics, Poincaré covariant BF sectors. In addition, we assume throughout this section that $(\mathcal{F}, \mathcal{U}(\tilde{\mathcal{P}}_+^\dagger), U(G), \Omega, k)$ satisfies the condition of geometric modular

action (as formulated in Proposition 7.5). We note that also in this case, as for localizable charges, this can be deduced from geometric modular action of the observable net, supplemented by mild additional assumptions [18, 23].

The considerations about the possible phase space behaviours of localizable charges discussed in Section 5 as a motivation for the preservice criterion for DHR-charges clearly apply also to the present case, as we are considering asymptotically localized field operators, for which an asymptotic phase space notion can be recovered. By this, we mean that if $\psi_1(\lambda), \dots, \psi_d(\lambda) \in \mathcal{F}(\lambda C)$ is some scaled multiplet associated to a fixed BF sector $[\rho]$ of the underlying theory, and if this multiplet is asymptotically localized in some $O \in \mathcal{K}$, then we can still think of the states $\psi_j(\lambda)\Omega$ as describing a charge $[\rho]$ which is, for small λ , essentially localized in λO so that, by looking at the energy content of these states, we can define their phase-space occupation. Furthermore, the direction of the cone C in which this multiplet is localized is irrelevant, in the sense that if $\hat{C} \supset O$ is another spacelike cone, we can find another multiplet $\hat{\psi}_1(\lambda), \dots, \hat{\psi}_d(\lambda) \in \mathcal{F}(\lambda \hat{C})$ still implementing the sector $[\rho]$. But from the picture of spacelike cones as strings which tend to vanish at small scales, we expect that, if also the multiplet $\hat{\psi}_j(\lambda)$ is asymptotically localized in O , then it should be possible to choose it in such a way that the charged states $\psi_j(\lambda)^*\Omega$, $\hat{\psi}_j(\lambda)^*\Omega$ should become close to each other as $\lambda \rightarrow 0$. This motivates the following generalization to the present setting of the notion of asymptotic containment, for whose formulation we introduce the notation $R_1 \Subset R_2$ for two arbitrary spacetime regions R_1, R_2 , to mean that there exists some neighbourhood of the identity $\mathcal{N} \subset \mathcal{P}_+^\uparrow$ such that $\mathcal{N} \cdot R_1 \subseteq R_2$.

Definition 8.1. Let $O_1 \in \mathcal{K}$, $C_1 \in \mathcal{S}$ be such that $O_1 \subset C_1$, and let $F \in \mathfrak{F}(C_1, O_1)$. F is said to be *asymptotically contained in* $\mathcal{F}_{0,\iota}(O)$ with $O \supset \overline{O_1}$, if for each spacelike cone $\hat{C}_1 \supset O_1$ there exist some $\hat{F} \in \mathfrak{F}(\hat{C}_1, O_1)$, with $\hat{F} = F$ for $\hat{C}_1 = C_1$, fulfilling the following properties:

- (A) $\lim_{\kappa} \left(\|(\hat{F}(\lambda_\kappa) - F(\lambda_\kappa))\Omega\| + \|(\hat{F}(\lambda_\kappa) - F(\lambda_\kappa))^*\Omega\| \right) = 0$;
- (B) for any $C \supset O$ such that $\hat{C}_1 \Subset C$, and for any $\epsilon > 0$, there exist elements $\underline{F}, \underline{F}' \in \mathfrak{F}(C, O)$ (depending on C and ϵ) so that

$$\limsup_{\kappa} \left(\|(\hat{F}(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})\Omega\| + \|(\hat{F}(\lambda_\kappa) - \underline{F}'_{\lambda_\kappa})^*\Omega\| \right) < \epsilon;$$

where $\{\lambda_\kappa\}_{\kappa \in K} \subset \mathbb{R}^+$ is such that $\omega_{0,\iota} = \lim_{\kappa} \omega_{\lambda_\kappa}$.

Lemma 8.2. Let $F \in \mathfrak{F}(C_1, O_1)$ and let $O \supset \overline{O_1}$. The following statements are equivalent:

- (a) F is asymptotically contained in $\mathcal{F}_{0,\iota}(O)$;
- (b) for each $\hat{C}_1 \supset O_1$ there exist some $\hat{F} \in \mathfrak{F}(\hat{C}_1, O_1)$, with $\hat{F} = F$ for $\hat{C}_1 = C_1$, fulfilling property (A) of Definition 8.1 and the following property
- (B') for any $C \supset O$ such that $\hat{C}_1 \Subset C$, and for any given $\epsilon > 0$ and finitely many $\underline{F}^{(1)}, \dots, \underline{F}^{(N)} \in \mathfrak{F}^\times$, there is $\underline{F} \in \mathfrak{F}(C, O)$ such that for $j = 1, \dots, N$;

$$\limsup_{\kappa} \left(\|(\hat{F}(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})\underline{F}_{\lambda_\kappa}^{(j)}\Omega\| + \|(\hat{F}(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})^*\underline{F}_{\lambda_\kappa}^{(j)}\Omega\| \right) < \epsilon;$$

(c) for each $\hat{C}_1 \supset O_1$ there exist some $\hat{F} \in \mathfrak{F}(\hat{C}_1, O_1)$, with $\hat{F} = F$ for $\hat{C}_1 = C_1$, such that

$$\lim_{(\kappa, \nu)} \left(\| ((\underline{\alpha}_{h_\nu} \hat{F})(\lambda_\kappa) - F(\lambda_\kappa)) \Omega \| + \| ((\underline{\alpha}_{h_\nu} \hat{F})(\lambda_\kappa) - F(\lambda_\kappa))^* \Omega \| \right) = 0 \quad (8.1)$$

whenever $\{h_\nu\}$ is a δ -sequence on $\tilde{\mathcal{P}}_+^\uparrow$, where the limit is taken with respect to the product partial ordering on $K \times \mathbb{N}$, and where

$$(\underline{\alpha}_h \hat{F})(\lambda) := \int_{\tilde{\mathcal{P}}_+^\uparrow} ds h(s) \alpha_{s_\lambda}(\hat{F}(\lambda)), \quad \lambda > 0, \quad h \in L^1(\tilde{\mathcal{P}}_+^\uparrow)$$

(integral in the weak sense, using the standard invariant measure on $\tilde{\mathcal{P}}_+^\uparrow$).

Proof. (a) \Rightarrow (c). The proof proceeds analogously to the proof of the corresponding implication in Lemma 5.3 using the estimate (with the notation introduced there)

$$\begin{aligned} \| ((\underline{\alpha}_{h_\nu} \hat{F})(\lambda_\kappa) - F(\lambda_\kappa))^\# \Omega \| &\leq 2 \| (\underline{F}_{\lambda_\kappa}^\# - \hat{F}(\lambda_\kappa))^\# \Omega \| \\ &\quad + \sup_{s \in \text{supp } h_\nu} \| \underline{\alpha}_s(\underline{F}^\#) - \underline{F}^\# \| + \| (\hat{F}(\lambda_\kappa) - F(\lambda_\kappa))^\# \Omega \|. \end{aligned}$$

(c) \Rightarrow (b). From the estimate

$$\begin{aligned} \| (\hat{F}(\lambda_\kappa) - F(\lambda_\kappa))^\# \Omega \| &\leq \sup_{s \in \text{supp } h_\nu} \| [(\mathcal{W}(s_{\lambda_\kappa}) - \mathbf{1}) \hat{F}(\lambda_\kappa)]^\# \Omega \| \\ &\quad + \| ((\underline{\alpha}_{h_\nu} \hat{F})(\lambda_\kappa) - F(\lambda_\kappa))^\# \Omega \|, \end{aligned}$$

and from strong continuity of $s \mapsto \mathcal{W}(s)$ property (A) for \hat{F} follows at once, by first choosing κ and then ν , depending on κ , sufficiently large. Property (B') is proven by the same argument as the one in the proof of the corresponding implication in Lemma 5.3, using here the fact that $\tilde{\mathcal{F}}_{0,\iota}^{\times t}(W) \Omega_{0,\iota} = V_{0,\iota}^\times \tilde{\mathcal{F}}_{0,\iota}^\times(W) \Omega_{0,\iota}$, with $V_{0,\iota}^\times$ the natural twisting operator on $\mathcal{H}_{0,\iota}^\times$, is dense in $\mathcal{H}_{0,\iota}^\times$, Theorem B.1.

(b) \Rightarrow (a). Obvious. \square

Remark. The field net on double cones defined by $\hat{\mathcal{F}}(O) := \bigcap_{C \supset O} \mathcal{F}(C)$ is essentially the net of the QFSGS determined by the localizable sectors of \mathcal{A}_{vac} (see [15] for precise statements), and we can associate to it scaling algebras $\hat{\mathfrak{F}}(O)$ and a scaling limit net $\hat{\mathcal{F}}_{0,\iota}(O)$ in the way discussed in previous sections. It is then clear that $\hat{\mathfrak{F}}(O) \subset \hat{\mathfrak{F}}(C, O)$ for each $C \supset O$ and that a function $F : \mathbb{R}^+ \rightarrow \hat{\mathfrak{F}}$ complying with properties (i)-(iii) in definition 5.2 is an element of $\mathfrak{F}(\hat{C}_1, O_1)$ for each $\hat{C}_1 \supset O_1$, so that if F is asymptotically contained in $\hat{\mathcal{F}}_{0,\iota}(O)$, it is also asymptotically contained in $\mathcal{F}_{0,\iota}(O)$ as it suffices to take $\hat{F} = F$ for each \hat{C}_1 in Definition 8.1. Conversely, if F with $F(\lambda) \in \hat{\mathcal{F}}(\lambda O_1)$ is asymptotically contained in $\mathcal{F}_{0,\iota}(O)$, then thanks to the characterizations (c) in Lemmas 5.3 and 8.2, it is also asymptotically contained in $\hat{\mathcal{F}}_{0,\iota}(O)$.

As in the case of localizable charges, a collection of multiplets $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda > 0}$ inducing a fixed BF sector $[\rho]$ and with $\psi_j(\lambda) \in \mathcal{F}(\lambda C)$ for some $C \in \mathbb{S}$, will be called a **scaled multiplet** for $[\rho]$.

Definition 8.3. Let $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}^\times}(\omega)$ be a scaling limit state of the underlying QFSGS, and let $[\rho] \in \text{Sect}_{\text{fin}}^{\text{BF}, \text{cov}}$ be a BF superselection sector. Then we say that the charge $[\rho]$ is *preserved in the scaling limit QFTGA* of $\omega_{0,\iota}$ if, for each $O_1 \in \mathcal{K}$, $C_1 \in \mathcal{S}$ with $C_1 \supset O_1$, there is some scaled multiplet $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ for $[\rho]$ such that all functions $\lambda \mapsto \psi_j(\lambda)$, $j = 1, \dots, d$, are elements of $\mathfrak{F}(C_1, O_1)$ and are asymptotically contained in $\mathcal{F}_{0,\iota}(O)$ if $O \supset \overline{O_1}$.

Proposition 8.4. Suppose that the charge $[\rho]$ is preserved in the scaling limit QFTGA of $\omega_{0,\iota}$. Let $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ be a scaled multiplet for $[\rho]$ such that $\psi_j(\cdot) \in \mathfrak{F}(C_1, O_1)$ is asymptotically contained in $\mathcal{F}_{0,\iota}(O)$. Let, for $j = 1, \dots, d$ and $\hat{C}_1 \supset O_1$, $\psi_j^{\hat{C}_1} \in \mathfrak{F}(\hat{C}_1, O_1)$ be as in Lemma 8.2(c). Then the limit operators

$$\psi_j = s\text{-}\lim_{\nu \rightarrow +\infty} \pi_{0,\iota}(\underline{\alpha}_{h_\nu} \psi_j^{\hat{C}_1}) \quad \text{and} \quad \psi_j^* = s\text{-}\lim_{\nu \rightarrow +\infty} \pi_{0,\iota}(\underline{\alpha}_{h_\nu} \psi_j^{\hat{C}_1})^* \quad (8.2)$$

exist for any δ -sequence $\{h_\nu\}$, are independent of \hat{C}_1 and of the chosen δ -sequence, and are contained in $\mathcal{F}_{0,\iota}(O)$ whenever $O \supset \overline{O_1}$. Furthermore, ψ_1, \dots, ψ_d forms a multiplet transforming under the adjoint action of $U_{0,\iota}^\bullet(G_{0,\iota}^\bullet)$ according to the irreducible, unitary representation $v_{[\rho]}$. More precisely, denoting by $G \ni g \mapsto g^\bullet \in G_{0,\iota}^\bullet$ the quotient map, there is a finite-dimensional, irreducible, unitary representation $v_{[\rho]}^\bullet$ of $G_{0,\iota}^\bullet$ so that $v_{[\rho]}^\bullet(g^\bullet) = v_{[\rho]}(g)$ for all $g \in G$ and

$$U_{0,\iota}^\bullet(g^\bullet) \psi_j U_{0,\iota}^\bullet(g^\bullet)^* = \sum_{i=1}^d \psi_i v_{[\rho]ij}^\bullet(g^\bullet), \quad g^\bullet \in G_{0,\iota}^\bullet.$$

Proof. The proof is similar to the proof of Proposition 5.5, so we will only indicate the major differences. From the inequality

$$\begin{aligned} & \| (\pi_{0,\iota}^\times(\underline{\alpha}_{h_\nu} \psi_j^{\hat{C}_1}) - \pi_{0,\iota}^\times(\underline{\alpha}_{\tilde{h}_\nu} \psi_j^{\tilde{C}_1})) \Omega_{0,\iota} \| \\ & \leq \limsup_{\kappa} \left(\| ((\underline{\alpha}_{h_\nu} \psi_j^{\hat{C}_1})(\lambda_\kappa) - \psi_j(\lambda_\kappa)) \Omega \| + \| ((\underline{\alpha}_{\tilde{h}_\nu} \psi_j^{\tilde{C}_1})(\lambda_\kappa) - \psi_j(\lambda_\kappa)) \Omega \| \right), \end{aligned}$$

valid for arbitrary δ -sequences $\{h_\nu\}$, $\{\tilde{h}_\nu\}$ and spacelike cones $\hat{C}_1, \tilde{C}_1 \supset O_1$, together with Lemma 8.2(c), it follows that $\lim_{\nu \rightarrow +\infty} \pi_{0,\iota}^\times(\underline{\alpha}_{h_\nu} \psi_j^{\hat{C}_1}) \Omega_{0,\iota} =: \Phi_j$ exists and is independent of \hat{C}_1 and of the chosen δ -sequence. As $\|\underline{\alpha}_{h_\nu} \psi_j^{\hat{C}_1}\|$ is uniformly bounded in ν , and since $\Omega_{0,\iota}$ is separating for $\tilde{\mathcal{F}}_{0,\iota}^\times(W)$ with $W \ni \hat{C}_1$, this implies that $s\text{-}\lim_{\nu \rightarrow +\infty} \pi_{0,\iota}^\times(\underline{\alpha}_h \psi_j^{\hat{C}_1}) =: \psi_j^\times$ exists and is defined by $\psi_j^\times f \Omega_{0,\iota} = f \Phi_j$ for $f \in \tilde{\mathcal{F}}_{0,\iota}^\times(W)'$. Then, since for any two spacelike cones \hat{C}_1, \tilde{C}_1 we can find spacelike cones $\hat{C}_2, \dots, \hat{C}_n$ with $\hat{C}_n = \tilde{C}_1$, and wedges W_1, \dots, W_{n-1} such that $\hat{C}_i \cup \hat{C}_{i+1} \Subset W_i$, $i = 1, \dots, n-1$, we conclude that ψ_j^\times is independent of \hat{C}_1 , as well as of $\{h_\nu\}$. Thus, since for any spacelike cone $C \supset O$ there is a $\hat{C}_1 \supset O_1$ such that $\hat{C}_1 \Subset C$, we have $\psi_j^\times \in \pi_{0,\iota}^\times(\mathfrak{F}(C, O))''$, and then $\psi_j := \psi_j^\times \upharpoonright \mathcal{H}_{0,\iota} \in \mathcal{F}_{0,\iota}(O)$. The same argument shows that $\pi_{0,\iota}(\underline{\alpha}_{h_\nu} \psi_j^{\hat{C}_1})^*$ converges strongly to ψ_j^* .

The rest of the proof is essentially identical to the corresponding part of the proof of Prop. 5.5. \square

If, for $F \in \mathfrak{F}(C_1, O_1)$, we define $\bar{F}(\lambda) := J_W V F(\lambda) V^* J_W$, where $W \supset C_1$ is a wedge and J_W is the associated modular conjugation, and recalling that we assume that the underlying QFSGS satisfies geometric modular action, it is easily checked that $\bar{F} \in \mathfrak{F}(r_W C_1, r_W O_1)$ and that $(\underline{\alpha}_h \bar{F})(\lambda) = J_W V(\underline{\alpha}_{h \circ \text{Adr}_W} F)(\lambda) J_W V^*$ (where $\widetilde{\text{Adr}_W}(L, a) := (\widetilde{\text{Adr}_W} L, r_W a)$), so that if F is asymptotically contained in $\mathcal{F}_{0,\iota}(O)$ then \bar{F} is asymptotically contained in $\mathcal{F}_{0,\iota}(r_W O)$, and it is then straightforward to verify that the following generalization of Theorem 5.8 holds.

Theorem 8.5. *Let $\omega_{0,\iota} \in \text{SL}^{\mathfrak{X}}(\omega)$ be a scaling limit state. Then a BF charge $[\rho] \in \text{Sect}_{\text{fin}}^{\text{BF}, \text{cov}}$ is preserved in the scaling limit state $\omega_{0,\iota}$ if and only if also the conjugate charge $[\bar{\rho}]$ is preserved.*

We would then like to obtain a result corresponding to Proposition 5.6. However, at the present stage of our work, this can be achieved only at the price of some additional assumptions on the net $\mathcal{F}_{0,\iota}$, namely that $\mathcal{A}_{0,\iota}(O) = \mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet}$. We will comment on this assumption below. Nevertheless, without making this assumption, we can at any rate show that the fields ψ_j constructed above give rise, in a sense made precise in the following theorem, to positive energy representations of $\mathfrak{A}_{0,\iota}$.

Theorem 8.6. *Let $\xi = [\rho]$ be a BF charge which is preserved in the scaling limit QFTGA of a given $\omega_{0,\iota}$, and let $\psi_j \in \mathcal{F}_{0,\iota}(O)$, $j = 1, \dots, d$ a multiplet for ξ arising as in proposition 8.4. Then the state ω_ξ on $\mathfrak{A}_{0,\iota}$ defined by*

$$\omega_\xi(\mathbf{a}) := \sum_{j=1}^d \langle \Omega_{0,\iota}, \psi_j \mathbf{a} \psi_j^* \Omega_{0,\iota} \rangle, \quad \mathbf{a} \in \mathfrak{A}_{0,\iota}, \quad (8.3)$$

obeys $\omega_\xi \upharpoonright \mathfrak{A}_{0,\iota}(O') = \omega_{0,\iota} \upharpoonright \mathfrak{A}_{0,\iota}(O')$ and induces, via the GNS construction, a representation π_ξ of $\mathfrak{A}_{0,\iota}$ which is locally normal and translation covariant.

Proof. It is evident that $\omega_\xi \upharpoonright \mathcal{A}_{0,\iota}(O)$ is a normal state, so that π_ξ is a locally normal representation of $\mathfrak{A}_{0,\iota}$. According to a theorem of Borchers [4, Thm. II.6.6], in order to show that π_ξ is translation covariant, it is necessary and sufficient to show that the set of vector states of π_ξ is contained in the norm closure of the set $\mathfrak{A}_{0,\iota}^*(V_+)$ of functionals $\phi \in \mathfrak{A}_{0,\iota}^*$ with the following property. For each pair $\mathbf{a}, \mathbf{b} \in \mathfrak{A}_{0,\iota}$, the function $x \in \mathbb{R}^4 \mapsto \phi(\mathbf{a} \alpha_x^{(0,\iota)}(\mathbf{b}))$ is continuous and is the boundary value of a function W which is analytic in the forward tube $\mathcal{T} := \mathbb{R}^4 + iV_+$ and satisfies the bound

$$|W(z)| \leq \|\mathbf{a}\| \|\mathbf{b}\| e^{m|\text{Im}z|}, \quad z \in \mathcal{T},$$

for some constant $m > 0$ which may depend on ϕ but not on \mathbf{a}, \mathbf{b} ; furthermore the same conditions must be satisfied by ϕ^* ($\phi^*(\mathbf{a}) := \overline{\phi(\mathbf{a}^*)}$).

Now, the set of operators $\mathbf{c} \in \mathfrak{A}_{0,\iota}$ with compact support in momentum space, i.e. where there exists a compact $\Delta \subset \mathbb{R}^4$ such that $\alpha_h^{(0,\iota)}(\mathbf{c}) = 0$ for each $h \in L^1(\mathbb{R}^4)$ with $\text{supp } \hat{h} \subset \mathbb{R}^4 \setminus \Delta$, is strongly dense in $\mathfrak{A}_{0,\iota}$. To see this, take $\mathbf{c} = \alpha_f^{(0,\iota)}(\mathbf{c}_1)$ with $\mathbf{c}_1 \in \mathfrak{A}_{0,\iota}$ and compact $\text{supp } \hat{f}$. Then \mathbf{c} has compact momentum space support. The set of L^1

functions f with compact supp \hat{f} is dense in L^1 . Owing to the fact that the action of $\alpha^{(0,\iota)}$ is strongly continuous on $\mathfrak{A}_{0,\iota}$, this implies that there exists a sequence of L^1 -functions f_n with compact supp \hat{f}_n so that $\alpha_{f_n}^{(0,\iota)}(\mathbf{c}_1)$ approaches c_1 in norm.

Then, using

$$\|\pi_\xi(\mathbf{c})\Omega_\xi - \pi_\xi(\mathbf{d})\Omega_\xi\|^2 = \sum_{j=1}^d \|(\mathbf{c} - \mathbf{d})\psi_j^*\Omega_{0,\iota}\|^2,$$

where Ω_ξ is a cyclic vector for π_ξ , it is sufficient to show that the functionals $\phi_{\mathbf{c}}(\mathbf{a}) := \omega_\xi(\mathbf{c}^*\mathbf{a}\mathbf{c})$, with $\mathbf{c} \in \mathfrak{A}_{0,\iota}$ having compact momentum space support, are contained in the norm closure of $\mathfrak{A}_{0,\iota}^*(V_+)$.

To this end let $\mathbf{c} \in \mathfrak{A}_{0,\iota}$ have momentum support in a compact set Δ , and let Δ_n be the closed double cone in momentum space with vertices 0 and $(n, \mathbf{0})$, $n > 0$. Then the functional $\phi_{\mathbf{c},n} \in \mathfrak{A}_{0,\iota}^*$ defined by

$$\phi_{\mathbf{c},n}(\mathbf{a}) := \sum_{j=1}^d \langle \Omega_{0,\iota}, \psi_j E(\Delta_n) \mathbf{c}^* \mathbf{a} \mathbf{c} E(\Delta_n) \psi_j^* \Omega_{0,\iota} \rangle,$$

where by E we denote the spectral measure associated to translations of $\mathcal{F}_{0,\iota}$, is such that the function $x \mapsto \phi_{\mathbf{c},n}(\mathbf{a} \alpha_x^{(0,\iota)}(\mathbf{b}))$ is continuous and has distributional Fourier transform with support in $-(\Delta + \Delta_n) + \overline{V}_+$. Then, if $p = (-m, \mathbf{0})$ is such that $-(\Delta + \Delta_n) + \overline{V}_+ \subset p + \overline{V}_+$, one concludes by [4, Thm. II.1.7] that this function is a boundary value of a function W analytic in \mathcal{T} and satisfies, for suitable constants $k, M, N > 0$, the bound

$$|W(z)| \leq k(1 + |x|)^N (1 + \text{dist}(y, \partial V_+)^{-1})^M e^{m|y|}, \quad z = x + iy \in \mathcal{T}.$$

But there also holds $|W(x)| \leq \|\phi_{\mathbf{c},n}\| \|\mathbf{a}\| \|\mathbf{b}\|$ for $x \in \mathbb{R}^4$, and this implies, by a standard analytic function argument,¹⁰ the desired estimate $|W(z)| \leq \|\phi_{\mathbf{c},n}\| \|\mathbf{a}\| \|\mathbf{b}\| e^{m|y|}$, showing that $\phi_{\mathbf{c},n} \in \mathfrak{A}_{0,\iota}^*(V_+)$. Then by the inequality

$$|\phi_{\mathbf{c}}(\mathbf{a}) - \phi_{\mathbf{c},n}(\mathbf{a})| \leq \sum_{j=1}^d 2\|\mathbf{c}\|^2 \|\mathbf{a}\| \| [E(\Delta_n) - \mathbf{1}] \psi_j^* \Omega_{0,\iota} \|,$$

we get the statement. \square

We denote by $\pi_{0,\iota}^{\text{vac}}$ the vacuum representation of $\mathfrak{A}_{0,\iota}$, defined by $\pi_{0,\iota}^{\text{vac}}(\mathbf{a}) := \mathbf{a} \upharpoonright \mathcal{H}_{0,\iota}^{\text{vac}}$, where $\mathcal{H}_{0,\iota}^{\text{vac}} = \overline{\mathfrak{A}_{0,\iota} \Omega_{0,\iota}}$ is the scaling limit vacuum Hilbert space. Thanks to the separating property of $\Omega_{0,\iota}$ for local algebras, $\pi_{0,\iota}^{\text{vac}} \upharpoonright \mathcal{A}_{0,\iota}(O)$ is an isomorphism of von Neumann algebras.

Corollary 8.7. *If the scaling limit vacuum Hilbert space $\mathcal{H}_{0,\iota}^{\text{vac}}$ is separable, then for each $x \in \mathbb{R}^4$*

$$\pi_\xi \upharpoonright \mathfrak{A}_{0,\iota}(O' + x) \cong \pi_{0,\iota}^{\text{vac}} \upharpoonright \mathfrak{A}_{0,\iota}(O' + x), \quad (8.4)$$

i.e. π_ξ has the DHR property for the class of all translates of the given double cone O .

¹⁰We are indebted to J. Bros for helpful remarks on this point.

Proof. By the argument in the appendix of [13], the fact that $\omega_\xi \upharpoonright \mathfrak{A}_{0,\iota}(O') = \omega_{0,\iota} \upharpoonright \mathfrak{A}_{0,\iota}(O')$, together with translation covariance of π_ξ , imply (8.4) if it is known that property B holds in the representation π_ξ . But if $\mathcal{H}_{0,\iota}^{\text{vac}}$ is separable, then each local algebra $\pi_{0,\iota}^{\text{vac}}(\mathcal{A}_{0,\iota}(O))$ has a separable predual, and being ω_ξ locally normal, from [35, corollary 3.2] it follows that the Hilbert space \mathcal{H}_ξ of π_ξ is separable, and then, by the already recalled argument of Roberts [31], property B holds in the representation π_ξ . \square

We recall that separability of $\mathcal{H}_{0,\iota}^{\text{vac}}$ follows from suitable nuclearity properties of the underlying observable net [7].

We now turn to discussing the conditions under which it is possible to generalize Proposition 5.6. At the technical level, the main obstruction is represented by the fact that, in general, $\mathcal{A}_{0,\iota}(O) \subsetneq \mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet}$, as it is easy to construct gauge invariant combinations of the $\underline{\alpha}_h \psi_j$'s which need not belong to some scaling algebra $\underline{\mathfrak{A}}(O)$ but are only localized in spacelike cones. However, thanks to the fact that these functions are asymptotically localizable in O , it may well happen that, at least in favourable cases, their scaling limits do belong to $\mathcal{A}_{0,\iota}(O)$. Adding the simple hypothesis that this is indeed the case, yields a quite satisfactory picture of the scaling limit of morphisms.

Proposition 8.8. *Let $\omega_{0,\iota} \in \text{SL}^{\mathfrak{F}^\times}(\omega)$ and assume that $\mathcal{A}_{0,\iota}(O) = \mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet}$ and that $\mathfrak{F}_{0,\iota}$ acts irreducibly on $\mathcal{H}_{0,\iota}$. Moreover, let $[\rho] \in \text{Sect}_{\text{fin}}^{\text{BF},\text{cov}}$ be a charge of the underlying QFSGS which is preserved in the scaling limit QFTGA of $\omega_{0,\iota}$, let $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ be a scaled multiplet for $[\rho]$ asymptotically contained in $\mathcal{F}_{0,\iota}(O)$ and let, with respect to this scaled multiplet, ψ_1, \dots, ψ_d be defined as in (8.2).*

If we define, for each $\underline{A} \in \underline{\mathfrak{A}}$, the family $\{\rho(\underline{A})(\lambda)\}_{\lambda>0}$ as

$$\rho(\underline{A})(\lambda) = \sum_{j=1}^d \psi_j(\lambda) \underline{A}_\lambda \psi_j(\lambda)^*,$$

then there holds

$$s\text{-}\lim_{\nu \rightarrow +\infty} \pi_{0,\iota}(\underline{\alpha}_{h_\nu} \rho(\underline{A})) = \sum_{j=1}^d \psi_j \pi_{0,\iota}(\underline{A}) \psi_j^*, \quad \underline{A} \in \underline{\mathfrak{A}}; \quad (8.5)$$

and ρ defined by

$$\rho(\underline{a}) = \sum_{j=1}^d \psi_j \underline{a} \psi_j^*, \quad \underline{a} \in \mathfrak{A}_{0,\iota}, \quad (8.6)$$

is a localized, transportable, irreducible endomorphism of $\mathfrak{A}_{0,\iota}$ which is moreover covariant and has finite statistics.

The proof of these statements is completely parallel to that of Proposition 5.6.

As a final comment, we would like to remark that the condition $\mathcal{A}_{0,\iota}(O) = \mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet}$, introduced here as a technical assumption in order to get a well defined scaling limit of morphisms, may turn out to have a sensible physical interpretation. By the above remarks, we see that $\mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet}$ contains, apart from the scaling limit observables localized in O , the scaling limit of functions $\lambda \mapsto \underline{A}_\lambda \in \mathcal{A}(\lambda C)$, for every spacelike cone $C \supset O$,

i.e. there are gauge invariant families of operators, with localization regions extending to spacelike infinity, which give rise to objects in the scaling limit which are charged with respect to the intrinsic gauge group of $\mathcal{A}_{0,\iota}$, so that new charges appear at small scales. This situation, which must not be confused with confinement where the fields carrying the new charges cannot be approximated at all at finite scales, is instead reminiscent of the phenomenon of *charge screening*,¹¹ much discussed in the physical literature (cf. for instance [32, 33] and references quoted). In this scenario, a charge which is described by an asymptotically free theory at small scales disappears at finite scales because, due to nonvanishing interactions, it is always accompanied by a cloud, extending to spacelike infinity, of charge-anticharge pairs, so that one can expect that the corresponding “charge carrying fields” are neutral and non-compactly localized at finite scales, and become instead charged and localized in the scaling limit. Then the condition $\mathcal{A}_{0,\iota}(O) = \mathcal{F}_{0,\iota}(O)^{G_{0,\iota}^\bullet}$ could be interpreted as the requirement that in the theory under consideration, no charges are screened.

Concluding Remarks

A generalization of the scaling algebra framework to the situation where the operator algebras describing the underlying quantum field theory contain charge-carrying fields has been developed in this work, together with a proposal as to what it means that a charge present in the underlying theory is preserved in the scaling limit. A natural concept of confined charge arises as a charge in the scaling limit theory which is not obtainable as a charge of the underlying theory which is preserved in the scaling limit process [6, 8]. We have indicated two basic physical mechanisms for the disappearance of charges in the scaling limit. Moreover, we have seen that the preservice of all charges in the scaling limit leads to the equivalence of local and global intertwiners for the superselection sectors in the underlying theory.

We hope that in the future it will be possible to illustrate the mechanisms for charge disappearance in the scaling limit by instructive examples, possibly in lower spacetime dimensions. This should also shed light on the very important issue if the lifted action of the gauge transformations shouldn’t be defined differently than in (2.3) in the case where the physical dimension of charge is related to the dimension of length. Furthermore, it would also appear desirable to develop, based on the method of scaling algebras for the observables of the underlying theory, an abstract renormalization group analysis for superselection charges without using the Doplicher-Roberts reconstruction theorem. This would eventually make superselection charges with braid group statistics and with infinite statistics accessible to short-distance analysis.

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¹¹This connection was pointed out to us by Detlev Buchholz.

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A An example of a preserved localizable charge

In this Appendix we shall show that the localizable charge described by the Majorana field in $n = 1 + 3$ spacetime dimensions with \mathbb{Z}_2 gauge group satisfies the preservice condition, Definition 5.4, in all scaling limit states.

For the definition of the Majorana field, we will mainly follow [19], where also a discussion of the superselection structure is given.

We begin with some notational conventions. Let

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \quad (\text{A.1})$$

be the Dirac matrices in chiral representation, where σ_j are the Pauli matrices. A vector $u \in \mathbb{C}^4$ (also called a spinor) will be thought as a column matrix and correspondingly its adjoint u^\dagger will be a row matrix, so that the standard scalar product on \mathbb{C}^4 is given by $(u, v) \mapsto u^\dagger v$ (rows by columns product of matrices). We adopt the notation $\not{v} := v_\mu \gamma^\mu$ for any (covariant) vector $v \in \mathbb{R}^4$. By Ω_m^\pm we shall indicate the upper and lower mass $m > 0$ hyperboloid, $\Omega_m^\pm := \{p \in \mathbb{R}^4 : p^2 = m^2, \pm p_0 > 0\}$.

For a given mass $m > 0$, the Dirac operator is $D := \gamma^\mu \partial_\mu + im$, and, denoting as usual by $\mathcal{D}(\mathbb{R}^4; \mathbb{C}^4)$ the space of spinor valued, compactly supported smooth functions on Minkowski space, we endow the space $H_{0,m} := \mathcal{D}(\mathbb{R}^4; \mathbb{C}^4) / \text{Im } D$ with the scalar product

$$\langle f, g \rangle_m := \int_{\mathbb{R}^3} d^3 \mathbf{p} \sum_{\pm} \hat{f}(\pm \omega_m(\mathbf{p}), \mathbf{p})^\dagger P_\pm(\mathbf{p}) \hat{g}(\pm \omega_m(\mathbf{p}), \mathbf{p}), \quad (\text{A.2})$$

where

$$P_\pm(\mathbf{p}) = \frac{\gamma^0(\not{p} + m)}{2p_0} \Big|_{p_0 = \pm \omega_m(\mathbf{p})}, \quad \omega_m(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2}, \quad (\text{A.3})$$

and where we made no notational distinction between elements in $H_{0,m}$ and their representatives in $\mathcal{D}(\mathbb{R}^4; \mathbb{C}^4)$. Let H_m be the completion of $H_{0,m}$ in this scalar product.

The action of the universal covering of the Poincaré group on H_m is defined, for $(A, a) \in \tilde{\mathcal{P}}_+^\uparrow$, by

$$(u(A, a)f)(x) := S(A)f(\Lambda(A)^{-1}(x - a)), \quad S(A) := \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix}, \quad (\text{A.4})$$

$A \in SL(2, \mathbb{C}) \mapsto \Lambda(A) \in \mathcal{L}_+^\uparrow$ being the covering homomorphism.

Let C be the antilinear operator on \mathbb{C}^4 defined by $Cu := i\gamma^2 \bar{u}$, where the bar denotes complex conjugation, which satisfies $C^2 = \mathbf{1}$, $C^\dagger = C$ and $C\gamma^\mu C = -\gamma^\mu$, and define then an antilinear involution Γ on $H_{0,m}$ by $(\Gamma f)(x) := Cf(x)$, which is antiunitary, $\langle \Gamma f, \Gamma g \rangle_m = \langle g, f \rangle_m$ (so that it extends to H_m) and commutes with the action of the Poincaré group.

Let $\mathfrak{B}(H_m)$ be the self-dual CAR algebra over H_m [1], generated as a C^* -algebra by elements $B(f)$, $f \in H_m$, such that $f \mapsto B(f)$ is antilinear, and

$$\{B(f), B(g)\} = \langle g, \Gamma f \rangle_m \mathbf{1}, \quad B(f)^* = B(\Gamma f). \quad (\text{A.5})$$

By CAR unicity, the representation u of $\tilde{\mathcal{P}}_+^\dagger$ on H_m induces an automorphic action α of $\tilde{\mathcal{P}}_+^\dagger$ on $\mathfrak{B}(H_m)$, defined by

$$\alpha_{(A,a)}(B(f)) := B(u(A,a)f), \quad (A,a) \in \tilde{\mathcal{P}}_+^\dagger, f \in H_m,$$

and, by the fact that $\|B(f)\| \leq 2\|f\|_m$ and strong continuity of u , it follows that this action is strongly continuous, i.e. $(A,a) \mapsto \alpha_{(A,a)}(B)$ is norm continuous for each $B \in \mathfrak{B}(H_m)$.

We consider on $\mathfrak{B}(H_m)$ the quasifree state ω defined, according to [2], by the 2-point function

$$\omega(B(f)B(g)) := \langle \Gamma f, P_+ g \rangle_m, \quad (\text{A.6})$$

where P_+ is the projection on the positive energy states in H_m , defined by $\widehat{P_+ f}(p_0, \mathbf{p}) = P_+(\mathbf{p})\hat{f}(p_0, \mathbf{p})$.

The action α of $\tilde{\mathcal{P}}_+^\dagger$ leaves ω invariant, so that if we consider the GNS representation $(\pi, \mathcal{H}, \Omega)$ induced by ω , we get on \mathcal{H} a unitary strongly continuous representation \mathcal{U} of $\tilde{\mathcal{P}}_+^\dagger$ leaving Ω invariant and such that (π, \mathcal{U}) is a covariant representation of $(\mathfrak{B}(H_m), \alpha)$.

Definition A.1. *The free Majorana field of mass $m > 0$ is the operator valued distribution $f \in \mathcal{D}(\mathbb{R}^4; \mathbb{C}^4) \mapsto \psi(f) \in \mathcal{B}(\mathcal{H})$ given by $\psi(f) := \pi(B(f))$, $f \in \mathcal{D}(\mathbb{R}^4; \mathbb{C}^4)$, where on the right hand side f is identified with its image in H_m .*

It is straightforward to verify that ψ is covariant with respect to \mathcal{U} ,

$$\mathcal{U}(A,a)\psi(f)\mathcal{U}(A,a)^* = \psi(u(A,a)f),$$

that the translations $a \mapsto \mathcal{U}(\mathbf{1}, a)$ satisfy the spectrum condition with Ω as the unique (up to a phase) translation invariant unit vector in \mathcal{H} , and that $\psi(f)$, $\psi(g)$ anticommute for spacelike separated $\text{supp} f$, $\text{supp} g$.

We now turn to the consideration of the net of local von Neumann algebras associated to the free Majorana field, and defined by

$$\mathcal{F}(O) := \{\psi(f) : \text{supp} f \subseteq O\}'', \quad (\text{A.7})$$

for $O \subset \mathbb{R}^4$ open and bounded. On this net the group \mathbb{Z}_2 acts by an automorphism β_k induced by the automorphism of $\mathfrak{B}(H_m)$ defined by $B(f) \mapsto -B(f)$, which leaves the vacuum state ω invariant, and is therefore implemented by a unitary operator $U(k)$ on \mathcal{H} such that $U(k)^2 = U(k^2) = \mathbf{1}$, so that it induces a direct sum decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ according to its eigenspaces, i.e. $U(k) \upharpoonright \mathcal{H}_\pm = \pm \mathbf{1}_{\mathcal{H}_\pm}$, which is Poincaré and gauge invariant. Define then $\mathcal{U}_{\text{vac}}(A,a) := \mathcal{U}(A,a) \upharpoonright \mathcal{H}_+$, and the net of observable von Neumann algebras associated to the free Majorana field as

$$\mathcal{A}_{\text{vac}}(O) := \mathcal{F}(O)^{\mathbb{Z}_2} \upharpoonright \mathcal{H}_+. \quad (\text{A.8})$$

That in this way we get an example satisfying the assumptions made in Sections 2 and 4 is the content of the following proposition, the proof of which, being standard, is omitted.

Proposition A.2. *With the above notations, and with $\mathcal{H}_{vac} := \mathcal{H}_+$, let π be the representation of the quasi-local algebra \mathfrak{A}_{vac} defined by $\pi(A \upharpoonright \mathcal{H}_{vac}) := A$. Then $(\mathcal{A}, \mathcal{U}_{vac}(\mathcal{P}_+^\dagger), \Omega)$ is a Poincaré covariant observable net, and $(\mathcal{F}, \mathcal{U}(\tilde{\mathcal{P}}_+^\dagger), U(\mathbb{Z}_2), \Omega, k)$ is a QFSGS on it.*

In the next proposition, the very simple superselection structure of \mathfrak{A}_{vac} described by the field net \mathcal{F} is analysed, cf. [19].

Proposition A.3. *The representation π_- of \mathfrak{A}_{vac} given by $\pi_- := \pi(\cdot) \upharpoonright \mathcal{H}_-$ satisfies the DHR criterion, is covariant, irreducible and with finite statistics, and any irreducible representation of \mathfrak{A}_{vac} appearing in \mathcal{H} is equivalent either to ι , the identity representation, or to π_- . Moreover, if for $f \in \mathcal{D}(O, \mathbb{C}^4)$ with $\text{supp} f \subseteq O$, $\Gamma f = f$ and $\|f\|_m = \sqrt{2}$, ρ_f is the automorphism of \mathfrak{A}_{vac} induced by the unitary operator $\psi(f) \in \mathcal{F}(O)$, then $\rho_f \in \Delta_{fin}^{cov}(O)$ and $\rho_f \cong \pi_-$.*

Proof. The irreducibility of π_- follows from the arguments in [12], taking into account that \mathcal{H}_- is the subspace associated to the irreducible representation $k \mapsto -1$ of \mathbb{Z}_2 in the factorial decomposition of U . This also implies that any other irreducible representation of \mathfrak{A}_{vac} in \mathcal{H} is equivalent to ι or π_- . To show that π_- satisfies the DHR criterion, fix a double cone O and an $f \in \mathcal{D}(O; \mathbb{C}^4)$ with $\Gamma f = f$ and $\|f\|_m = \sqrt{2}$. Then $\psi(f)$ is unitary by the CARs, and $\psi(f)\mathcal{H}_\pm = \mathcal{H}_\mp$. Let then $V_f := \psi(f) \upharpoonright \mathcal{H}_-$, and $g, h \in \mathcal{D}(O', \mathbb{C}^4)$. We have $V_f \pi_-(\psi(g)\psi(h)) = \psi(g)\psi(h)V_f$, so that V_f intertwines between $\pi_- \upharpoonright \mathfrak{A}_{vac}(O')$ and $\iota \upharpoonright \mathfrak{A}_{vac}(O')$. Covariance of π_- follows by $\mathcal{U}_- := \mathcal{U}(\cdot) \upharpoonright \mathcal{H}_-$. Finally if $\pi \rho_f(A) = \psi(f)\pi(A)\psi(f)^*$ then ρ_f is localized in O and has finite statistics, and V_f intertwines between π_- and ρ_f , and if $\text{supp} f_1 \subseteq O_1$, $V := \psi(f)\psi(f_1)^* \upharpoonright \mathcal{H}_+$ intertwines between ρ_{f_1} and ρ_f , which is therefore transportable. \square

Finally we come to the proof of the fact that the charge $\xi := [\pi_-]$ in the above proposition is preserved in any scaling limit theory, in the sense of Definition 5.4. To this end it is sufficient to find, for every double cone O_1 , a family $(f_\lambda)_{\lambda \in (0,1]}$ of functions such that $\text{supp} f_\lambda \subseteq \lambda O_1$, $\|f_\lambda\|_m = \sqrt{2}$, and such that condition (5.4) is satisfied for $\psi(f_\lambda)$.

Proposition A.4. *For every double cone O_1 , there exists $f \in \mathcal{D}(O_1; \mathbb{C}^4)$ such that, if $f_\lambda \in \mathcal{D}(\lambda O_1; \mathbb{C}^4)$ is defined by*

$$f_\lambda(x) := \lambda^{\frac{3}{2}-4} f(\lambda^{-1}x), \quad \lambda \in (0, 1],$$

then $\|f_\lambda\|_m = \sqrt{2}$ and $\Gamma f_\lambda = f_\lambda$ for $\lambda \in (0, 1]$, and $\lambda \mapsto \psi(\lambda) := \psi(f_\lambda)$ is asymptotically contained in $\mathfrak{F}_{0,\iota}(O)$ for each $O \supset \overline{O}_1$ and each scaling limit state $\omega_{0,\iota}$.

In the course of the proof of this proposition, we will need the following simple result concerning the action of the Lorentz group on Minkowski space.

Lemma A.5. *Fix a mass $m > 0$. For any sufficiently large $R > 0$, there exists a neighbourhood of the identity \mathcal{N} in \mathcal{L}_+^\dagger such that, for any $p \in \overline{V}_+$ with $0 \leq p^2 \leq m^2$ and $|\mathbf{p}| > R$, and for any $\Lambda \in \mathcal{N}$, it holds, for $p' := \Lambda p$, $|\mathbf{p}'| > |\mathbf{p}|/\sqrt{2}$.*

Proof. To simplify the notation, we will write $\Lambda \cdot \mathbf{p}$ for the spatial part, in a given Lorentz frame, of the 4-vector Λp , $\Lambda \in \mathcal{L}_+^\uparrow$, $p \in \mathbb{R}^4$. Let $\Lambda_1(s)$, $s \in \mathbb{R}$, denote the 1-parameter group of boosts in the p_1 direction.

If $p \in \mathbb{R}^4$ is such that $|p_1|^2 \leq |p_2|^2 + |p_3|^2$, since $\Lambda_1(s)$ leaves the components p_2, p_3 unaffected, we have, for any $s \in \mathbb{R}$, $|\Lambda_1(s) \cdot \mathbf{p}|^2 \geq |p_2|^2 + |p_3|^2 \geq |\mathbf{p}|^2/2$.

Assume now that $|p_1|^2 > |p_2|^2 + |p_3|^2$. This implies $|p_1| \geq |\mathbf{p}|/\sqrt{2}$ and, since for any sufficiently large $R > 0$,

$$\inf_{\substack{0 \leq p^2 \leq m^2 \\ |\mathbf{p}| > R}} \frac{|\mathbf{p}|}{|p_0|} \geq \inf_{|\mathbf{p}| > R} \frac{|\mathbf{p}|}{\sqrt{|\mathbf{p}|^2 + m^2}} > 0,$$

we can find a $\delta > 0$ such that, if $|s| < \delta$, $|(\Lambda_1(s)p)_1| = |\sinh s p_0 + \cosh s p_1| \geq |p_1|/\sqrt{2}$ for any $p \in \overline{V}_+$ with $0 \leq p^2 \leq m^2$ and $|\mathbf{p}| > R$, so that $|\Lambda_1(s) \cdot \mathbf{p}|^2 \geq p_1^2/2 + p_2^2 + p_3^2 \geq |\mathbf{p}|^2/2$.

Then, if we identify in the canonical way $SO(3)$ with a subgroup of \mathcal{L}_+^\uparrow , we conclude with $\mathcal{N} := \{R_1 \Lambda_1(s) R_2 : |s| < \delta, R_1, R_2 \in SO(3)\}$. \square

Proof of Proposition A.4. In order to shorten formulae, we will use the notation $p_{\lambda, \pm} := (\pm \omega_{\lambda m}(\mathbf{p}), \mathbf{p})$, as well as the notation $\Lambda \cdot \mathbf{p}$ introduced in the proof of the above lemma. Also, $|\cdot|$ will denote the norm of a vector both in \mathbb{R}^3 and in \mathbb{C}^4 . A calculation shows

$$\|f_\lambda\|_m^2 = \|f\|_{\lambda m}^2 = \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{4\omega_{\lambda m}(\mathbf{p})^2} \sum_{\pm} |\gamma^0 (\not{p}_{\lambda, \pm} + \lambda m) \hat{f}(p_{\lambda, \pm})|^2,$$

and then, in order to show that there is an $f \in \mathcal{D}(O, \mathbb{C}^4)$ such that $\|f_\lambda\|_m = \sqrt{2}$ and $\Gamma f_\lambda = f_\lambda$ for each $\lambda \in (0, 1]$, it is sufficient to exhibit an $f \in \mathcal{D}(O, \mathbb{C}^4)$ such that $\Gamma f = f$, and for which $(\not{p} + \mu) \hat{f}(p)$ is not identically zero on each hyperboloid $\Omega_\mu := \Omega_\mu^+ \cup \Omega_\mu^-$, $\mu > 0$. A direct check shows that these conditions are met by $f(x) := g(x)(\mathbf{1} + i\gamma^2) \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^t$ where $g \in \mathcal{D}(O; \mathbb{R})$.

We now show that actually $\lambda \mapsto \psi(\lambda)$ is an element of the scaling algebra itself, i.e. that

$$\lim_{(A, a) \rightarrow (1, 0)} \sup_{\lambda \in (0, 1]} \|\alpha_{(A, \lambda a)}(\psi(f_\lambda)) - \psi(f_\lambda)\| = 0, \quad (\text{A.9})$$

which clearly implies the statement. We have

$$\begin{aligned} \|\alpha_{(A, \lambda a)}(\psi(f_\lambda)) - \psi(f_\lambda)\|^2 &\leq 4 \|u(A, a)f - f\|_{\lambda m}^2 \\ &= 4 \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{4\omega_{\lambda m}(\mathbf{p})^2} \sum_{\pm} \left| \gamma^0 (\not{p}_{\lambda, \pm} + \lambda m) \left(e^{ip_{\lambda, \pm} \cdot a} S(A) \hat{f}(\Lambda(A)^{-1} p_{\lambda, \pm}) - \hat{f}(p_{\lambda, \pm}) \right) \right|^2, \end{aligned}$$

and, considering only the $+$ term in the sum inside the integral (the other one is estimated

in the same way), and writing $p_\lambda := p_{\lambda,+}$,

$$\begin{aligned}
& \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{4\omega_{\lambda m}(\mathbf{p})^2} \left| \gamma^0 (\not{p}_\lambda + \lambda m) \left(e^{ip_\lambda \cdot a} S(A) \hat{f}(\Lambda(A)^{-1} p_\lambda) - \hat{f}(p_\lambda) \right) \right|^2 \\
& \leq \frac{5}{4} \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{|\mathbf{p}|} |e^{ip_\lambda \cdot a} S(A) \hat{f}(\Lambda(A)^{-1} p_\lambda) - \hat{f}(p_\lambda)|^2 \\
& \leq \frac{5}{4} \left\{ \|S(A)\| \left[\|\hat{f}(\Lambda(A)^{-1} p_\lambda) - \hat{f}(p_\lambda)\|_2 + \|(e^{ip_\lambda \cdot a} - 1) \hat{f}(p_\lambda)\|_2 \right] \right. \\
& \quad \left. + \|S(A) - \mathbf{1}\| \|\hat{f}(p_\lambda)\|_2 \right\}^2,
\end{aligned} \tag{A.10}$$

where $\|\cdot\|_2$ denotes the standard norm in $L^2(\mathbb{R}^3, d^3 \mathbf{p}/|\mathbf{p}|) \otimes \mathbb{C}^4$, and where, for more clarity, we indicated explicitly the variable of integration inside the norms. The last term of the last line in this equation can be estimated uniformly in λ by the fact that, being $\hat{f} \in \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4)$, there are constants $C > 0$, $n > 1$, such that

$$\int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{|\mathbf{p}|} |\hat{f}(p_\lambda)|^2 \leq C \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{|\mathbf{p}|} \frac{1}{(1 + \omega_{\lambda m}(\mathbf{p})^2 + |\mathbf{p}|^2)^n} \leq C \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{|\mathbf{p}|} \frac{1}{(1 + 2|\mathbf{p}|^2)^n},$$

so that it can be made arbitrarily small, as $A \rightarrow \mathbf{1}$, uniformly in $\lambda \in (0, 1]$. For the second term in square brackets at the end of (A.10), we have, by an application of Lagrange's theorem to the exponential,

$$\int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{|\mathbf{p}|} |(e^{ip_\lambda \cdot a} - 1) \hat{f}(p_\lambda)|^2 \leq C(|a^0|^2 + |\mathbf{a}|^2) \int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{|\mathbf{p}|} \frac{|\omega_m(\mathbf{p})|^2 + |\mathbf{p}|^2}{(1 + 2|\mathbf{p}|^2)^n},$$

and then, if $n > 2$, this term is also uniformly small in the relevant limit. Finally, we use the above lemma to estimate the first term in square bracket at the end of (A.10). For each sufficiently large $R > 0$ let \mathcal{N}_R be a neighbourhood of the identity in $SL(2, \mathbb{C})$, such that $\Lambda(\mathcal{N}_R) \subseteq \mathcal{L}_+^\uparrow$ is as in the lemma. Then, for $|\mathbf{p}| > R$, $A \in \mathcal{N}_R$,

$$\begin{aligned}
|\hat{f}(\Lambda(A)^{-1} p_\lambda) - \hat{f}(p_\lambda)| & \leq C \left[\frac{1}{(1 + 2|\Lambda(A)^{-1} \cdot \mathbf{p}_\lambda|^2)^n} + \frac{1}{(1 + 2|\mathbf{p}_\lambda|^2)^n} \right] \\
& \leq C \left[\frac{1}{(1 + |\mathbf{p}|^2)^n} + \frac{1}{(1 + 2|\mathbf{p}|^2)^n} \right].
\end{aligned}$$

Thus, again by Lagrange theorem, we have, for $A \in \mathcal{N}_R$,

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{d^3 \mathbf{p}}{|\mathbf{p}|} |\hat{f}(\Lambda(A)^{-1} p_\lambda) - \hat{f}(p_\lambda)|^2 & \leq C \int_{|\mathbf{p}| > R} \frac{d^3 \mathbf{p}}{|\mathbf{p}|} \left[\frac{1}{(1 + |\mathbf{p}|^2)^n} + \frac{1}{(1 + 2|\mathbf{p}|^2)^n} \right]^2 \\
& \quad + \|\partial \hat{f}\|_\infty^2 \|\Lambda(A)^{-1} - \mathbf{1}\|^2 \int_{|\mathbf{p}| < R} \frac{d^3 \mathbf{p}}{|\mathbf{p}|} (|\omega_m(\mathbf{p})|^2 + |\mathbf{p}|^2).
\end{aligned}$$

and the λ independent right hand side can be made arbitrarily small by taking R sufficiently large, and A in a corresponding neighbourhood $\tilde{\mathcal{N}}_R \subseteq \mathcal{N}_R$. \square

B Reeh-Schlieder property for $\tilde{\mathfrak{F}}_{0,\iota}^\times(W)$

We employ the notations introduced in Section 7. Let $\tilde{\mathfrak{F}}_{0,\iota}^\times(C)$ be the C^* -algebra generated by $\pi_{0,\iota}^\times(\mathfrak{F}(C, O))$ as $O \subset C$.

Theorem B.1. *The vacuum $\Omega_{0,\iota}$ is a cyclic and separating vector for the algebras $\tilde{\mathfrak{F}}_{0,\iota}^\times(W)$.*

Let $\tilde{\mathfrak{F}}_{0,\iota}^\times(C)$ be the C^* -algebra generated by $\pi_{0,\iota}^\times(\mathfrak{F}(C, O))$ as $O \subset C$.

We will give a sketch of the proof of this theorem, which uses in an essential way analyticity of both translations and Lorentz boosts, consequence of geometric modular action and Tomita-Takesaki theory. Similar results can be found in [5, 16], to which we refer the interested readers for the details, which can also be found in [26].

We need some preparations. We recall that we denote by $\Lambda_W(t) \in \mathcal{P}_+^\uparrow$, $t \in \mathbb{R}$, the one parameter group of Poincaré transformations leaving the wedge W invariant. In order to simplify notations, we will identify $\Lambda_W(t)$ with its unique smooth lift to $\tilde{\mathcal{P}}_+^\uparrow$ which is the identity for $t = 0$.

Lemma B.2. *Let $\mathcal{U}_{0,\iota}^\times$ be a strongly continuous unitary representation of $\tilde{\mathcal{P}}_+^\uparrow$, and $\mathcal{N} \subseteq \tilde{\mathcal{P}}_+^\uparrow$ an open neighbourhood of the identity. Then $\mathcal{U}_{0,\iota}^\times(\tilde{\mathcal{P}}_+^\uparrow)$ is the strong closure of the group $\mathcal{U}_{\mathcal{N}}$ generated by the elements $\mathcal{U}_{0,\iota}^\times(s\Lambda_W(t)s^{-1})$, $t \in \mathbb{R}$, $s \in \mathcal{N}$.*

Proof. We can assume that $W = W_R$, and that $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \subseteq SL(2, \mathbb{C}) \times \mathbb{R}^4$. Then by [5, lemma 2.1], $\mathcal{U}_{0,\iota}^\times(SL(2, \mathbb{C}))$ is the strong closure of the subgroup of $\mathcal{U}_{\mathcal{N}}$ generated by $\mathcal{U}_{0,\iota}^\times(\Lambda\Lambda_{W_R}(t)\Lambda^{-1})$, $t \in \mathbb{R}$, $\Lambda \in \mathcal{N}_1$,¹² so that it is sufficient to show that $\mathcal{U}_{0,\iota}^\times(x) \in \mathcal{U}_{\mathcal{N}}^-$. Furthermore, as $\mathcal{U}_{0,\iota}^\times((\Lambda, x)\Lambda_{W_R}(t)(\Lambda, x)^{-1})\mathcal{U}_{0,\iota}^\times(\Lambda\Lambda_{W_R}(t)\Lambda^{-1})^* = \mathcal{U}_{0,\iota}^\times(\Lambda(\mathbf{1} - \Lambda_{W_R}(t))\Lambda^{-1}x)$, and $\mathcal{U}_{0,\iota}^\times(x) = \mathcal{U}_{0,\iota}^\times(x/n)^n$, $x \in \mathbb{R}^4$, $n \in \mathbb{N}$, we reduce the problem to showing that the set $E := \{\sum_i \Lambda_i(\mathbf{1} - \Lambda_{W_R}(t_i))\Lambda_i^{-1}x_i : (\Lambda_i, x_i) \in \mathcal{N}, t_i \in \mathbb{R}\}$ is a neighbourhood of zero in \mathbb{R}^4 . Then with e_μ , $\mu = 0, \dots, 3$ the canonical basis of \mathbb{R}^4 and $e_\pm := e_1 \pm e_0$, it is easily verified, by first choosing the Λ_i in the definition of E to be $\mathbf{1}$ and then to be a small rotation around the e_3 axis, that $se_\alpha \in E$ for $|s|$ sufficiently small and $\alpha = +, -, 2, 3$, so that E contains a neighbourhood of 0. \square

Lemma B.3. *The state $\omega_{0,\iota} = \langle \Omega_{0,\iota}, (\cdot)\Omega_{0,\iota} \rangle$ is a (-2π) -KMS state for the C^* -dynamical system $(\tilde{\mathfrak{F}}_{0,\iota}^\times(W), \alpha_{\Lambda_W}^{(0,\iota)\times})$.*

The proof is completely analogous to the one of the first part of Lemma 6.2 in [10]. For any finite set of spacelike cones C_1, \dots, C_n , we introduce the C^* -algebra $\tilde{\mathfrak{F}}_{0,\iota}^\times(C_1, \dots, C_n)$ as the one generated by the algebras $\tilde{\mathfrak{F}}_{0,\iota}^\times(C_1), \dots, \tilde{\mathfrak{F}}_{0,\iota}^\times(C_n)$. We also define $\mathfrak{G}_{0,\iota}^\times(C_1, \dots, C_n)$ to be the set of operators $G \in \tilde{\mathfrak{F}}_{0,\iota}^\times(C_1, \dots, C_n)$ for which there exists a neighbourhood \mathcal{N} of the identity in $\tilde{\mathcal{P}}_+^\uparrow$ such that $\alpha_s^{(0,\iota)\times}(G) \in \tilde{\mathfrak{F}}_{0,\iota}^\times(C_1, \dots, C_n)$ for any $s \in \mathcal{N}$. It is clear that $\mathfrak{G}_{0,\iota}^\times(C_1, \dots, C_n)$ is a $*$ -algebra and that for any n -tuple $\tilde{C}_1, \dots, \tilde{C}_n$ with $\tilde{C}_i \Subset C_i$, $i = 1, \dots, n$, $\tilde{\mathfrak{F}}_{0,\iota}^\times(\tilde{C}_1, \dots, \tilde{C}_n) \subseteq \mathfrak{G}_{0,\iota}^\times(C_1, \dots, C_n)$.

¹²the cited results refers actually to representations of \mathcal{L}_+^\uparrow , but since the proof uses only properties of its Lie algebra, it can be also applied to the present case

Lemma B.4. *Let W be a wedge in Minkowski space and let $C_i \in W$, $i = 1, \dots, n$ be spacelike cones. If $\Phi \in (\mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)\Omega_{0,\ell})^\perp$, then*

$$\langle \Phi, \alpha_{s_1}^{(0,\ell)\times}(G_1) \dots \alpha_{s_m}^{(0,\ell)\times}(G_m)\Omega_{0,\ell} \rangle = 0 \quad (\text{B.11})$$

for any $s_i \in \tilde{\mathcal{P}}_+^\uparrow$, $G_i \in \mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)$, $i = 1, \dots, m$.

Proof. We begin by showing that $\Phi \in (\mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)\Omega_{0,\ell})^\perp$ implies

$$\mathcal{U}_{0,\ell}^\times(s)\Phi \in (\mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)\Omega_{0,\ell})^\perp, \quad s \in \tilde{\mathcal{P}}_+^\uparrow.$$

Let \mathcal{N} be a neighbourhood of the identity in $\tilde{\mathcal{P}}_+^\uparrow$ such that $\mathcal{N}^{-1} \cdot C_i \subset W$, $i = 1, \dots, n$, and let $G \in \mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)$. Then there exists $\varepsilon > 0$, depending on $s \in \mathcal{N}$, such that $\langle \Phi, \alpha_{s\Lambda_W(t)s^{-1}}^{(0,\ell)\times}(G)\Omega_{0,\ell} \rangle = 0$ for $|t| < \varepsilon$. But by the above lemma $t \mapsto \alpha_{s\Lambda_W(t)s^{-1}}^{(0,\ell)\times}(G)\Omega_{0,\ell}$ has an analytic continuation to a function on the strip $\{-\pi < \text{Im } z < 0\}$, so that $\mathcal{U}_{0,\ell}^\times(s\Lambda_W(t)s^{-1})\Phi \in (\mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)\Omega_{0,\ell})^\perp$ for any $t \in \mathbb{R}$, $s \in \mathcal{N}$. Then, iterating the argument, and using lemma B.2, $\mathcal{U}_{0,\ell}^\times(s)\Phi \in (\mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)\Omega_{0,\ell})^\perp$ for each $s \in \tilde{\mathcal{P}}_+^\uparrow$.

If we now show that, for any $G_i \in \mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)$, $s_i \in \tilde{\mathcal{P}}_+^\uparrow$, $i = 1, \dots, m$, we have $\alpha_{s_1}^{(0,\ell)\times}(G_1) \dots \alpha_{s_m}^{(0,\ell)\times}(G_m)\Phi \in (\mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)\Omega_{0,\ell})^\perp$, since $\mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)$ is a $*$ -algebra containing the identity operator, the conclusion of the lemma will follow, but this is proven easily by induction, using the first part of the proof. \square

Proof of Theorem B.1. By normal commutation relations, it is sufficient to show that $\Omega_{0,\ell}$ is cyclic for $\tilde{\mathcal{F}}_{0,\ell}^\times(W)$, i.e. $(\tilde{\mathcal{F}}_{0,\ell}^\times(W)\Omega_{0,\ell})^\perp = \{0\}$. Let then $\Phi \in (\tilde{\mathcal{F}}_{0,\ell}^\times(W)\Omega_{0,\ell})^\perp$ and $F_i \in \tilde{\mathcal{F}}_{0,\ell}^\times(\tilde{C}_i)$, $i = 1, \dots, n$, be arbitrary operators. For any $i = 1, \dots, n$ there exists $s_i \in \tilde{\mathcal{P}}_+^\uparrow$ and a spacelike cone C_i such that $s_i^{-1} \cdot \tilde{C}_i \in C_i \in W$. Then $\alpha_{s_i^{-1}}^{(0,\ell)\times}(F_i) \in \mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)$ and, being $\Phi \in (\mathfrak{G}_{0,\ell}^\times(C_1, \dots, C_n)\Omega_{0,\ell})^\perp$,

$$\langle \Phi, F_1 \dots F_n \Omega_{0,\ell} \rangle = \langle \Phi, \alpha_{s_1}^{(0,\ell)\times}(\alpha_{s_1^{-1}}^{(0,\ell)\times}(F_1)) \dots \alpha_{s_n}^{(0,\ell)\times}(\alpha_{s_n^{-1}}^{(0,\ell)\times}(F_n))\Omega_{0,\ell} \rangle = 0$$

by lemma B.4, thus Φ is orthogonal to a total set of vectors in $\mathcal{H}_{0,\ell}^\times$, and then vanishes. \square

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